Systemic Behaviour of Plane Reciprocal Frame Structures

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Abstract

Reciprocal frame construction refers to a structural system that was developed first in eastern Asia in the 12th century. Short bar-shaped elements allow a surface to be spanned whose area is many times that of the length of the individual bars. In addition to the global geometry of the resulting surface, of particular interest is the interaction of forces between the individual bars that enables the load support and gives rise to the specific systemics of the overall structure. This paper intends to analyse these topics and the resulting possibilities.

Keywords: Discrete structures; bar-shaped structures; spatial structures; structural analysis; analysis method.

Introduction

A reciprocal frame is understood as a structural system formed by a number of short bars that are connected using friction only and span many times the length of the individual bars (*Fig. 1a*). This paper describes an academic view of the structural behaviour of such systems. A method is presented that describes the distribution of forces through the structure and can serve as a basis of a design method for practical usage.

History

The first reciprocal frame structures appeared in Chinese and Japanese architecture in the 12th century, primarily with wood-constructed roof support system[1] described as the mandala roof (*Fig. 1b*). Today, this roof design and also more complex reciprocal frame designs (*Fig. 1d*) are mainly used in Japanese architecture (see, e.g., Kazuhiro Ishii[2] and Shigeru Ban[3]).

In Europe, reciprocal frame structures were first introduced in the 13th century by the gothic architect Villard de Honnecourt.[4] His sketch books show illustrations of suggestions for roof support systems using this design principle. In the beginning of the 16th century, Leonardo da Vinci developed idea sketches for bridges, roofs and ceilings according to the principle of reciprocal frames,[5] and Sebastiano Serlio addressed the problem of how to span a ceiling with beams that were significantly shorter than the span of the ceiling itself (*Fig. 2a*).[6] A comparable structure system made of reciprocally supporting bar-shaped elements is the Zollinger System which is mainly used in timber roof constructions.[7] Friedrich Zollinger obtained a patent for it in 1923.

Sophisticated timber products—such as glulam trusses and plywood—that produce long spanning structural elements through adhesive technology have led to the replacement of reciprocal frames and similar structures.

However, these highly interesting supporting structures can be found today in academic environment where the system continues to be used as an experimental model due to the fascinating simplicity of the reciprocally bearing elements. In addition, research interest is motivated by the fact that comparable structures of discrete elements do occur in nature, for example in bird nests. In the course of the ever-increasing significance of biomorphic architectural language, it is immediately important to become acquainted with the functional behaviour of such structures.[8]

In the course of awareness of recyclability and resource saving, reciprocal

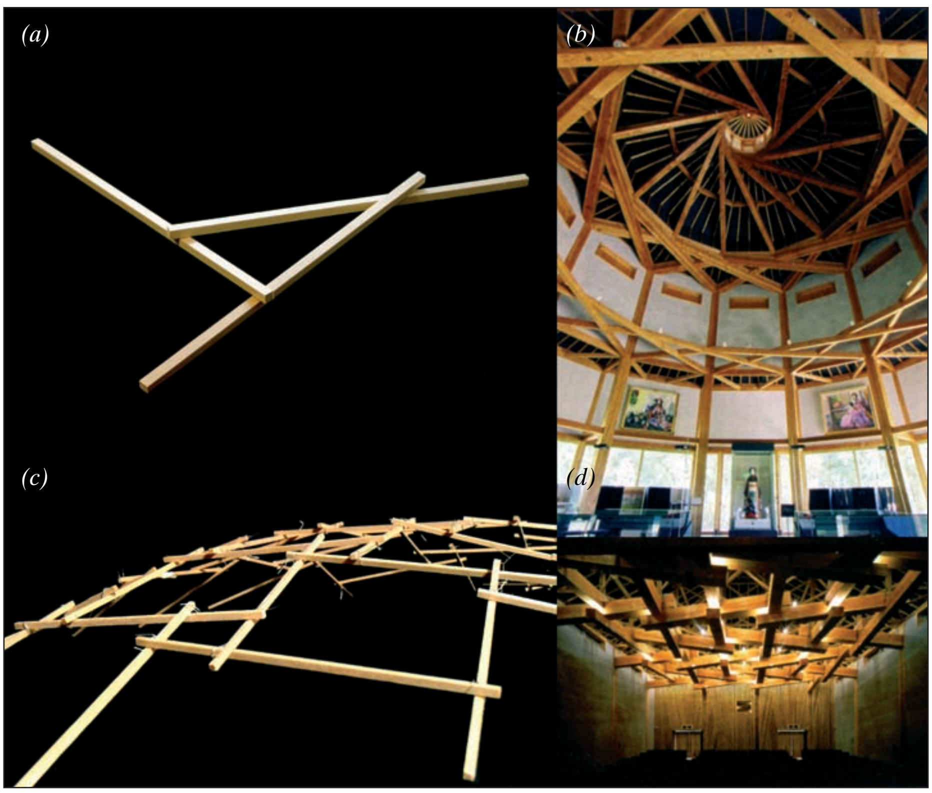

Fig. 1: (a) Simple reciprocal frame structure; (b) Bunraku Puppet Theatre—Exhibition Hall, Seiwa, Kazuhiro Ishii, 1994[2] (S. 93); (c) more complex reciprocal frame structure[2] (S. 23); (d) Bunraku Puppet Theatre—Auditorium, Seiwa, Kazuhiro Ishii, 1994[2] (S. 101)

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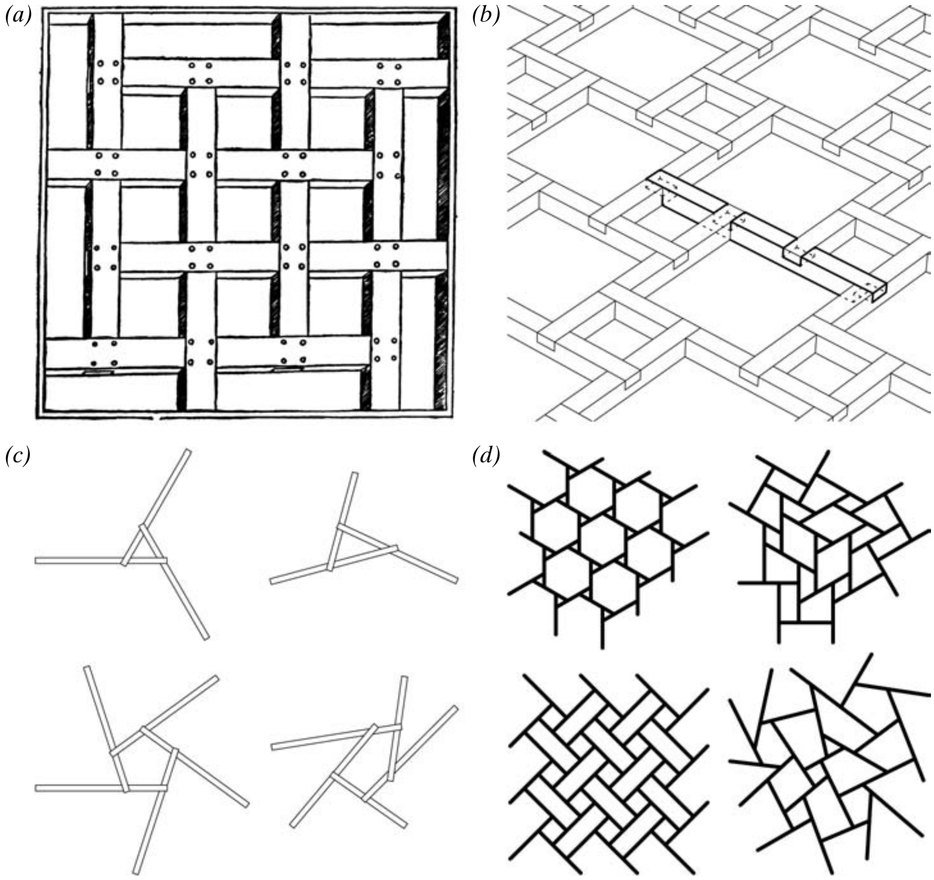

Fig. 2: (a) Sebastiano Serlios proposal of a ceiling structure composed of short elements[6] (S. 31); (b) plane reciprocal frame structure; (c) examples of basic elements: Regular (left), irregular (right); (d) examples of reciprocal frame structures consisting of identical basic elements (left), consisting of non-identical basic elements (right)

frames and similar structures should be considered again. If the internal forces are known, each element in such structures can be adjusted to its stress and therefore an optimized material consumption can be guaranteed. Nowadays, such customization is easily realizable through the possibility of digital fabrication.

Geometry and Load-Bearing Behaviour

A reciprocal frame structure is decomposable into basic elements which circumscribe a polygon with at least three sides whereby the figure may be either regular or irregular (*Fig. 2c*). A reciprocal frame structure can be constructed from identical or non-identical basic elements as long as a tessellation pattern exists (*Fig. 2d*).

The joining of the elements at the node points can generally be carried out without mechanical connections, but solely by pressure and friction. To support the frictional force, simple connection techniques such as tying together (*Fig. 1c*) or notching of the bars at the contact points may be used (*Fig. 2b*). Directly dependent on the development of the connections is the deformation of the entire system under loading. Increased slippage that occurs with simple joining, such as tying together, results in increased deformability of the entire structure.

From a structural standpoint, each individual bar in the system functions as a single beam. This beam lies at each of the bar's ends either on another bar or, if it forms the edge of the system, on the supports of the entire system. Each bar bears the supporting force of the one or two bars resting on it and optional dead loads or live loads (*Fig. 7*).

With straight bars lying on top of each other at their node points, the height offset between the contact points of the acting forces and the supporting loads creates a convex curvature of the system (*Fig. 1c*). In this case, the degree of curvature can be controlled by the gradients of bar length and the distance between the acting force and the supporting force. Additional options for controlling the gradients exist in the form of bending of the bars or notching at the node points, whereby gradient values ≤ 0 can be achieved. A gradient <0 produces a concave curvature, and a gradient $= 0$ of each bar results in a plane system (*Fig. 2b*).

In a horizontal plane reciprocal frame structure, the individual bars can be assumed to be statically determinate sub-systems, due to the fact that they act as a simple beam. The supporting forces of each bar are thus independent of the material and bar cross section.

With regard to horizontal forces, a three-bar basic element is statically determinate internally to the extent that the intersections are assumed to be flexible connections. Higher barred elements or elements consisting of more than three bars are moveable. The static determinacy of the supporting structure in regard to horizontal load can be controlled by appropriate combinations of higher barred basic elements and by appropriately supporting the entire structure.

Systemic Observation

In the following text, the interaction of the elements of reciprocal frame structures under loading will be addressed. Only plane structures under perpendicular loading will be considered. The exclusion of spatial structures and non-perpendicular loads is justified in that the methods shown here must have the basic prerequisite of decomposability of the entire system into a statically determinant component system.

Distinctions shall be made between the following hierarchical systems:

- Single bar (a in *Fig. 3*): smallest system unit.
- Basic system (b in *Fig. 3*): decomposable into single bars.
- Component system (c in *Fig. 3*): decomposable into basic systems and single bars.
- Entire system (*Fig. 3*): decomposable into component systems, basic systems and single bars.

The individual systems can be combined with each other. This results in hierarchically equal or higher system types. The connection within the system is defined by node points.

In this view, the load of a system is seen not as a static equilibrium state in which acting forces are established from the supporting forces, but the load is considered to be an iterative process of the interaction of sub-systems. In this connection, each iteration step is representative of the static equilibrium state observation of sub-systems. The

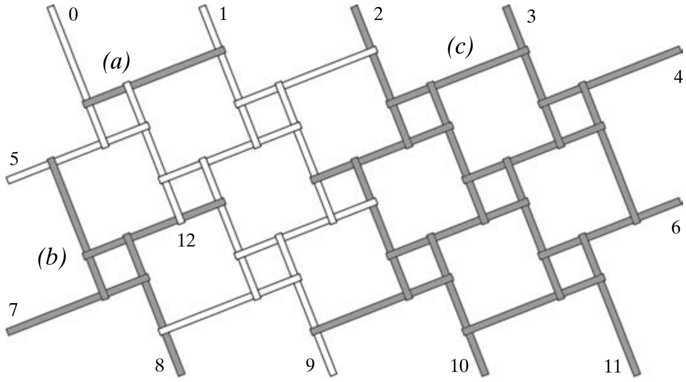

Fig. 3: Entire system with sub-systems. (a) top left: Single bar; (b) bottom left: Basic system; (c) right: component system

initial state of the iteration process is the external load of the system.

In each iteration step, the supporting forces from the conditions of equilibrium result from the progressions of the observed sub-system. These in turn represent in the next iteration step the progressions of the observed neighbouring system. As the process progresses, the number of observed sub-systems increases (a in *Fig. 4*).

Beginning at any starting point or at any iteration step, the analysis of the iteration step progressions leads to a differentiation of two different types of iteration step progressions (b in *Fig. 4*).

Cyclical: Progression where any bar is involved in the observation recurrently each time after a specific number of iteration steps. A bar in a cyclical progression thus demonstrates a concatenation of interactions with itself. The entirety of bars observed within one cyclical progression is defined as one possible basic system.

Diffused: Progression where each bar is involved in the observation only once, thus no bar exhibits a concatenation of interactions with itself. Such progressions describe the dispersion of the iteration steps in the system and describe the linking of the basic system with its neighbouring system.

Behaviour of a Basic System

First of all, the iterative process will be analysed on a basic system, based on a decomposition into sub-systems. Here, the sub-systems are single bars, representing the smallest system unit of a reciprocal frame structure.

In the following, n is the number of individual bars constituting the basic system whereby n must be >3, as otherwise no operating system is possible. Furthermore, k is the index of the observed bar and i that of the present iteration step. $K_{A,k}$ and $K_{B,k}$ are the node points of the system, whereby $K_{A,k}$ forms the support points on the system edge and $K_{B,k}$ the contact points to the neighbouring bar. A_k and B_k are the respective nodal forces. a_k and b_k apply to the proportions of node distances on the single bar, k.

$$a_k = \frac{\overline{K_{B,k-1}K_{B,k}}}{\overline{K_{A,k}K_{B,k}}} \leq 1$$

$$b_k = \frac{\overline{K_{A,k}K_{B,k-1}}}{\overline{K_{A,k}K_{B,k}}} \leq 1 \qquad (1)$$

$$a_k + b_k = 1$$

The condition of equilibrium results in the sub-system of the single bar k from the acting force B_{k-1}:

$$A_k = a_k \cdot B_{k-1}$$

$$B_k = b_k \cdot B_{k-1} \qquad (2)$$

As it is a matter of cyclical observation, the index $k = -1$ is the equivalent of $k = n-1$, and $k = n$ is the equivalent of $k = 0$. Therefore, the variables $u = (k + i)\mathrm{mod}\,n$ and $v = (k + i - 1)\mathrm{mod}\,n$ are introduced. Thus, if B_k is the acting force on the basic system, A_u and B_u apply to the increase in static forces at points $K_{A,u}$ and $K_{B,u}$ in any iteration step $i > 0$.

$$A_u = a_u \cdot B_v$$

$$B_u = b_u \cdot B_v \qquad (3)$$

Each bar of the basic system is involved exactly once in an equilibrium formulation within a progression of n iteration steps, whereby the node forces A_k and B_k receive a value each time, which is the increase in static force at nodes $K_{A,k}$ and $K_{B,k}$ in each iteration step. *Figure 6* illustrates the values B_k in a regular four-barred basic system with one acting force at node point $K_{B,3}$. It shows an exponentially decreasing increase in static forces at the nodes.

The equilibrium formulation of all node points can be described in one term through matrix notation. Thus, one step of the iterative process for any basic system made up of n bars can generally be formulated as follows:

$$\mathbf{F}_i = \mathbf{A}\mathbf{F}_{i-1} \qquad (4)$$

where $\mathbf{A}$, denoted in the following as the load distribution matrix, includes the geometrical conditions of all bars in a planar basic system (e.g. see *Fig. 5*).

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_0 \\ 0 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-1} & 0 & 0 \end{pmatrix}$$

$$\mathbf{A} \in \mathbb{R}^{2n \times 2n} \qquad (5)$$

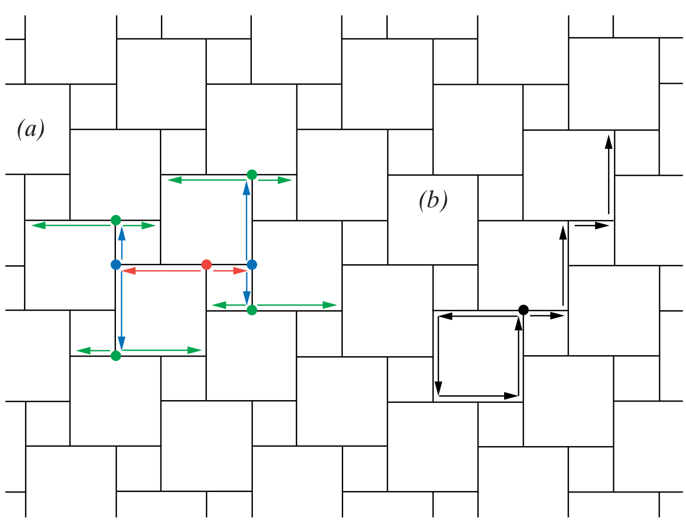

Fig. 4: (a) left: Example of three iteration steps: first red, second blue, third green; (b) right: Example of a cyclical and a diffused iteration step progression

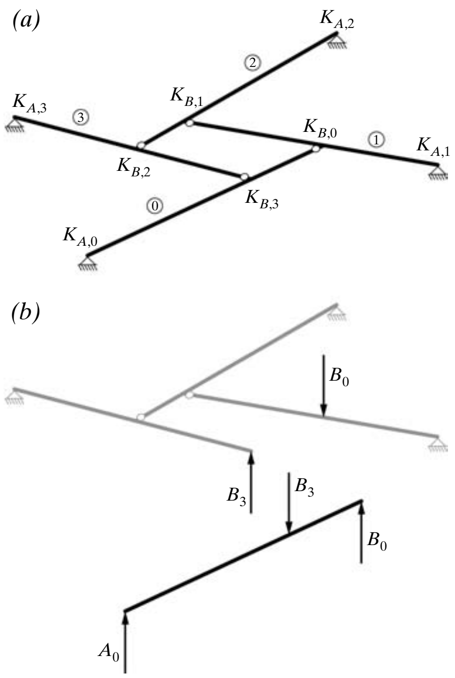

Fig. 5: Example of a basic system (n = 4). (a) statics system; (b) sub-system "bar 0"

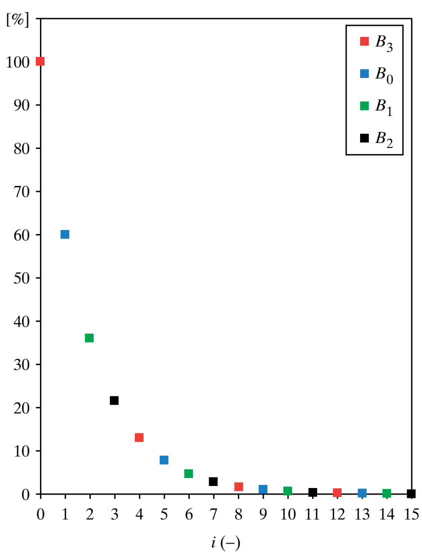

Fig. 6: Iteration process of B_k with $n = 4$, $a_k = 0{,}4$, $b_k = 0{,}6$

$\mathbf{F}_i$ (in Eq. 4) includes all the increases in node forces in the observed iteration step i:

$$\mathbf{F}_i = \begin{pmatrix} A_{0,i} \\ B_{0,i} \\ A_{1,i} \\ B_{1,i} \\ \cdots \\ \cdots \\ A_{n-1,i} \\ B_{n-1,i} \end{pmatrix}$$

$$\mathbf{F} \in \mathbb{R}^{2n} \tag{6}$$

$\mathbf{F}_0$ relates to the beginning of the iterative observation, and comprises the external forces of the system. It applies to the increase of node forces in iteration step i:

$$\mathbf{F}_i = \mathbf{A}^i\mathbf{F}_0 \tag{7}$$

With the maximum norm $\|\mathbf{A}\| < 1$ applies

$$\lim_{i\to\infty} \mathbf{F}_i = \mathbf{0} \tag{8}$$

$\mathbf{F}_t$ applies to the node forces existing in iteration step t. These are the sum of all force increases of the preceding iteration steps:

$$\mathbf{F}_t = \sum_{i=0}^{t} \mathbf{F}_i \tag{9}$$

$\mathbf{F} = \lim_{t\to\infty} \mathbf{F}_t$ results with the maximum norm $\|\mathbf{A}\| < 1$:

$$\mathbf{F} = \sum_{i=0}^{\infty} \mathbf{F}_i = \left(\sum_{i=0}^{\infty} \mathbf{A}^i\right)\mathbf{F}_0 = (\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{F}_0 \tag{10}$$

identity matrix $\mathbf{E} \in \mathbb{R}^{2n\times 2n}$

The node forces $\mathbf{F}$ determined in this manner (Eq. 10) correspond to the values of a static equilibrium observation in the classical sense.

As an example, a three-barred basic system as seen in *Fig. 2c* is considered. The proportions of the node distances in all bars are $a = 0{,}3$ and $b = 0{,}7$. This results in a load distribution matrix as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0{,}3 \\ 0 & 0 & 0 & 0 & 0 & 0{,}7 \\ 0 & 0{,}3 & 0 & 0 & 0 & 0 \\ 0 & 0{,}7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0{,}3 & 0 & 0 \\ 0 & 0 & 0 & 0{,}7 & 0 & 0 \end{pmatrix} \tag{11}$$

For the external load, only dead load is considered. Therefore the load in each node point is half of the bar weight, $g = 0{,}5 \cdot S$. With it applies

$$\mathbf{F}_0 = \begin{pmatrix} 0{,}5\cdot S \\ 0{,}5\cdot S \\ 0{,}5\cdot S \\ 0{,}5\cdot S \\ 0{,}5\cdot S \\ 0{,}5\cdot S \end{pmatrix} \tag{12}$$

The limit of the iterative observation as shown above (Eq.10) results in the following:

$$\mathbf{F} = \begin{pmatrix} S \\ 1{,}667\cdot S \\ S \\ 1{,}667\cdot S \\ S \\ 1{,}667\cdot S \end{pmatrix} \tag{13}$$

The components $A_k = S$ (support points of the system) and $B_k = 1{,}667 \cdot S$ (contact points to the neighbouring bars) included in $\mathbf{F}$ are equivalent to the node forces resulting from a static observation of this system in the classical sense.

Systemic structure of reciprocal frame structures

To comprehend the iteration process of an entire reciprocal frame structure as described above, it is necessary to formulate the load distribution matrix $\mathbf{A}$ of the entire system. Using a node-wise process, as shown in the previous section, can become very complex.

For this reason, the following shows the analogies between hierarchical structures of systems according to the definition given at the beginning and according to the structure of the load distribution matrix $\mathbf{A}$ from sub-matrices. This information allows for the systematic creation of the load distribution matrix for a complex reciprocal frame system.

The structure takes place according to the hierarchy defined above.

Single Bar

When observing an individual bar of a reciprocal frame system, four node points can be specified where forces may occur:

- Two points $K_{F,0}$ and $K_{F,3}$, where the bar lies on its neighbouring bars or with the support of the entire structure.
- Two points $K_{F,1}$ and $K_{F,2}$, which in turn form the support for other bars and, as a result, acting forces can result at these points.

a, b, c and *d* apply to the proportions of node distances of a single bar (*Fig. 7*).

$$a = \frac{\overline{K_{F,2}K_{F,3}}}{\overline{K_{F,0}K_{F,3}}} \le 1 \quad b = \frac{\overline{K_{F,0}K_{F,2}}}{\overline{K_{F,0}K_{F,3}}} \le 1 \tag{14}$$

$$a + b = 1$$

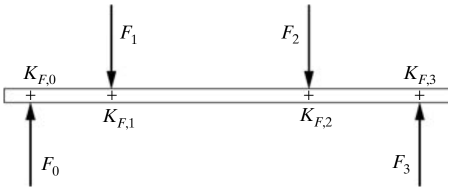

Fig. 7: Node points and forces of a single bar

$$c = \frac{\overline{K_{F,1}K_{F,3}}}{\overline{K_{F,0}K_{F,3}}} \leq 1 \quad d = \frac{\overline{K_{F,0}K_{F,1}}}{\overline{K_{F,0}K_{F,3}}} \leq 1 \qquad (14)$$

$$c + d = 1$$

Analogous to what was shown in the previous section, an iteration step can be described as follows:

$$\mathbf{F}_{i,S} = \mathbf{A}_S^i \mathbf{F}_{0,S}$$

with

$$\mathbf{A}_S = \begin{pmatrix} 0 & c & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d & b & 0 \end{pmatrix}$$

$$\mathbf{F}_{0,S} = \begin{pmatrix} F_{0,0} \\ F_{1,0} \\ F_{2,0} \\ F_{3,0} \end{pmatrix} \qquad (15)$$

Since the system is a statically determinate system, the elements of the load distribution matrix $\mathbf{A}_s$ consist only of geometrical factors. $\mathbf{F}_{0,s}$ comprises the node forces at the beginning of the iterative observation.

$\mathbf{F}_{i,s}$ applies to the single bar for $i > 1$.

$$\mathbf{F}_{i,S} = \mathbf{0}$$

Basic System

The basic system is built from n single bars. The following intends to demonstrate that the system's load distribution matrix $\mathbf{A}_G$ can be set up from a structure of $n \times n$ sub-matrices.

For that purpose, we must note in a first step the n load distribution matrices $\mathbf{A}_{s,k}$ of the single bars on the main diagonals of the basic system's matrix.

$$\mathbf{A}_G^* = \begin{pmatrix} \mathbf{A}_{S,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{S,1} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{S,n-1} \end{pmatrix} \qquad (16)$$

As only the main diagonals are being used, there is no interaction of the sub-matrices in this matrix structure. This corresponds to a system made up of n independent bars (*Fig. 8a*).

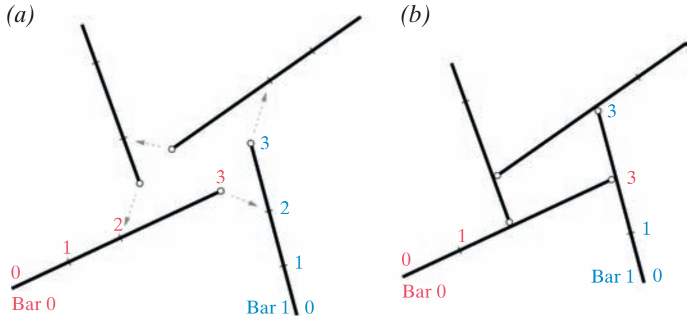

Fig. 8: Example n = 4: (a) System with independent bars; (b) system with dependent bars

To achieve a system with dependent bars (*Fig. 8b*), sub-matrices must be produced in the load distribution matrix on the other side of the main diagonals that describe the dependence of the single bars. These sub-matrices are described in the following text.

The connecting of two bars corresponds to the overlaying of two node points. This results in a merging together of the two columns assigned to the nodes listed in the load distribution matrix $\mathbf{A}_G^*$.

The assignment of node g on bar k in the system to column h of matrix $\mathbf{A}_G^*$ is

$$h = 4 \cdot (k - 1) + g \qquad (17)$$

Should any node g_0 from bar k_0 be overlaid with node g_1 from bar k_1, this corresponds to the transposition of corresponding columns h_0 and h_1 of $\mathbf{A}_G^*$ (*Fig. 9a*), as well as to the subsequent removal of column h_1 and of line h_1 (*Fig. 9b*).

As an example, this overlaying of two nodes is shown in a basic system made up of four bars (*Fig. 9*).

If this method is used to proceed with all four connection nodes, the basic system results (*Fig. 8b*), which correspond to the following load distribution matrix:

$$\mathbf{A}_G = \begin{pmatrix}
\cdot & c_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & d_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_0 \\
\cdot & \cdot & a_1 & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & b_1 & \cdot & d_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & a_2 & \cdot & c_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & b_2 & \cdot & d_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_3 & \cdot & c_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_3 & \cdot & d_3 & \cdot
\end{pmatrix} \qquad (18)$$

$\mathbf{A}_G$ (Eq. 18) is functionally identical to the load distribution matrix $\mathbf{A}$ of the basic system shown in the previous section (Eq. 5). However, in $\mathbf{A}_G$ the non-loaded node points are taken into consideration and in the following become significant when the basic system is expanded.

Component System

In the same way that a basic system is constructed from single bars, the formation of a suitable component system is made up of identical or non-identical basic systems. An observation

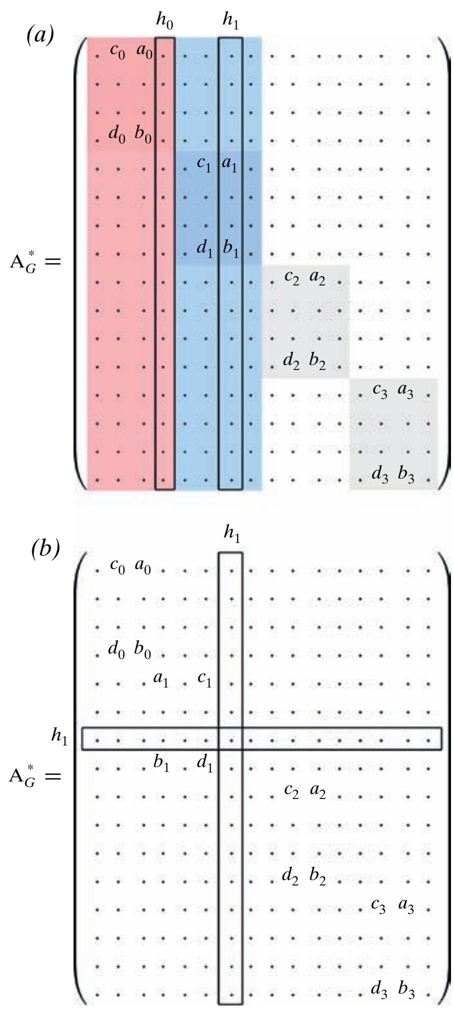

Fig. 9: (a) Transposition of columns h_0 and h_1; (b) removal of column h_1 and line h_1

of component systems is useful if patterns repeat in a reciprocal frame structure.

To comprehend a component system made up of m basic systems, the load distribution matrices of the basic systems have to be placed on the main diagonals of the component system's matrix. This corresponds to m independent basic systems (*Fig. 10a*).

$$\mathbf{A}_T^* = \begin{pmatrix} \mathbf{A}_{G,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{G,1} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{G,m-1} \end{pmatrix} \quad (19)$$

To establish a connection of the basic system (*Fig. 10b*), sub-matrices must be produced on the other side of the main diagonals of $\mathbf{A}_T^*$ which describe the interaction of the diagonal elements. This takes place analogous to the process that was described for the construction of the basic system.

As an example, the following is the load distribution matrix results for a component system made up of four basic systems:

$$\mathbf{A}_T = \begin{pmatrix} \mathbf{A}^*_{G,0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{C}_0 \\ \mathbf{C}_1 & \mathbf{A}^*_{G,1} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{A}^*_{G,2} & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{C}_3 & \mathbf{A}^*_{G,3} \end{pmatrix} \quad (20)$$

For this component system, two different types of sub-matrices result. **B** describes the connection to the basic element joined in a counter-clockwise manner; **C** is the connection to the basic element which joins in a clockwise manner.

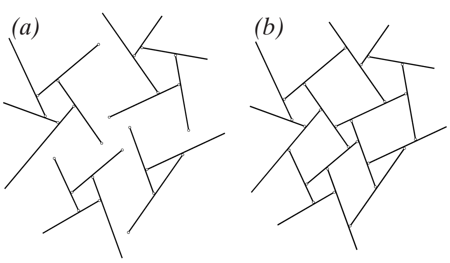

Fig. 10: (a) Component system with independent basic systems; (b) component system with dependent basic systems

For systemic observation, distinctions were made at the beginning of the text between cyclical and diffused progressions of iterative steps. The load distribution matrix can be likewise decomposed into a cyclical part $\mathbf{A}_{T,z}$ and a diffused part $\mathbf{A}_{T,d}$. It is valid

$$\mathbf{A}_{T,z} = \begin{pmatrix} \mathbf{A}^*_{G,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^*_{G,1} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^*_{G,m-1} \end{pmatrix}$$

$$\mathbf{A}_{T,d} = \mathbf{A}_T - \mathbf{A}_{T,z} \quad (21)$$

These parts, for the example shown above, are

$$\mathbf{A}_{T,z} = \begin{pmatrix} \mathbf{A}^*_{G,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^*_{G,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^*_{G,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}^*_{G,4} \end{pmatrix} \quad (22)$$

$$\mathbf{A}_{T,d} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \mathbf{C}_1 \\ \mathbf{C}_2 & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_3 & \mathbf{0} & \mathbf{B}_3 \\ \mathbf{B}_4 & \mathbf{0} & \mathbf{C}_4 & \mathbf{0} \end{pmatrix}$$

In each of the basic systems from which the component system is made, one cyclical progression of iteration steps can be determined. This progression is described by the cyclical part $\mathbf{A}_{T,z}$. Each sub-matrix $\mathbf{A}^*_{G,p}$ $(0 \le p < m)$ here describes those progressions of iteration steps that comprise exclusively components of the accompanying subsystem p.

$\mathbf{A}_{T,d}$ describes the diffusion of the iterative observation, and thus those progressions of iterative steps that describe the interaction of the individual sub-systems.

The main matrix $\mathbf{A}_T$ and its diagonal elements $\mathbf{A}^*_{G,p}$ must be quadratic; however, the sub-matrices of the diffused part must not.

Entire System

In the same manner that component systems are constructed from basic systems, entire systems can be built from basic and component systems. For this, the results of the load distribution matrix **A**, which regarding construction and characteristics, is comparable to the load distribution matrix of a component system.

A is also decomposable into a cyclical part $\mathbf{A}_z$ and a diffused part $\mathbf{A}_d$, both of which behave similar to parts $\mathbf{A}_{T,z}$ and $\mathbf{A}_{T,d}$ for a component system.

In order to verify the method of iterative observation of element interaction, the results of different structures were compared with those of a static calculation through the finite element software Cubus. As an example, the comparison for the entire system shown in *Fig. 3* is illustrated here. It is loaded with a force of 100 kN at node 12 and has proportions of node distances of a = 0,3 and b = 0,7 in all bars. Table 1 shows the results of the supporting points (nodes 0 to 11) in both procedures. It can be seen that the results differ from each other by less than 1%.

Conclusion

In this paper, a possibility has been described to examine the forces within plane reciprocal frame structures. The illustrated approach is not meant to be a practical design procedure but is to be considered as merely academic. In the first place, it is to develop an understanding of the behaviour of such structures and it can therefore form the basis of practical methods that analyse and design reciprocal frames or comparable structures.

The method is a systematic analysis of the interactive behaviour of the totality of all sub-systems in a plane reciprocal frame structure. Analogous to construction from sub-systems, construction of the load distribution matrix **A**, which is necessary for the observation shown here, is a result of sub-matrices. As such, sub-systems and reciprocal frame structures can be of any form if construction of the bars is geometrically compatible and the structure can function according to the reciprocal frame principle.

By using the iterative observation method to observe the increase in node

Node no.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Iterative (kN)	6,63	13,32	8,4	2,04	0,44	18,15	0,82	13,48	20,33	13,14	2,76	0,49
Finite Element Method (kN)	6,57	13,4	8,39	2,02	0,44	18,22	0,82	13,35	20,4	13,12	2,78	0,49
Difference (%)	0,9	0,6	0,12	0,98	0	0,39	0	0,96	0,34	0,15	0,72	0

Table 1: Evaluation of the proposed method by the Finite Element Method

forces within the system, it is possible to illustrate the systemic behaviour of the interaction of sub-systems in reciprocal frame structures as follows:

$$\mathbf{F}_i = \mathbf{A}^i \mathbf{F}_0 \tag{7}$$

By illustrating the individual iteration steps, it is possible to produce a simulation comparable to the load distribution in a structure. Moreover, the limit of the sum of all iteration steps can be uniformly formulated as follows:

$$\mathbf{F} = (\mathbf{E} - \mathbf{A})^{-1} \mathbf{F}_0 \tag{10}$$

This limit **F** consists of all node forces of the structure according to the static equilibrium.

A comparable systemic observation can also be applied to various other structures. However, the prerequisite exists that the structures must be made up of discrete elements and that the sub-system of each element must be assumed to be a statically determinate equilibrium system.

For instance, this method can be used to determine the lateral forces at the node points of grillages, assuming that all loads act perpendicular to the grillage. A decomposability into elements must be found that interact on the principle of reciprocal frames, so that each sub-system (*Fig. 11*) can then be assumed to be a statically determinate equilibrium system. In an easy manner, the internal forces of a grid can be found through this method and, out of this, simple geometrical rules for the formation of the individual bars can be established so that they meet the static demands.

In the illustrated example (*Fig. 12*), the individual steps of an iterative observation of sub-systems on a grillage are shown starting at the acting force F at the node point K_F. The figure shows the smallest possible number of iteration steps for each of the four support points $K_{A,w}$ $(0 \leq w < 4)$ up to the point that it is involved in the observation for the first time. Since the forces observed in each step decrease exponentially as the iteration progresses (*Fig. 6*), the method demonstrates that the forces in a load-bearing structure are primarily carried by the support which exhibits the smallest distance to the acting force.

If the structural conditions of a slab that is being acted upon by perpendicular forces are illustrated with appropriate grillage, this method can also be applied.

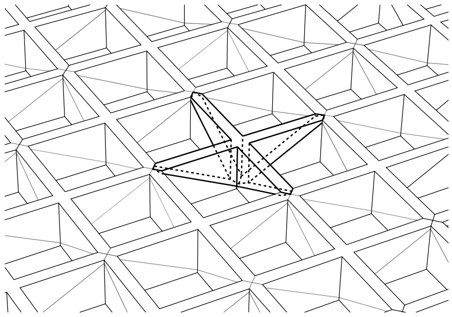

Fig. 11: Elements of a grid

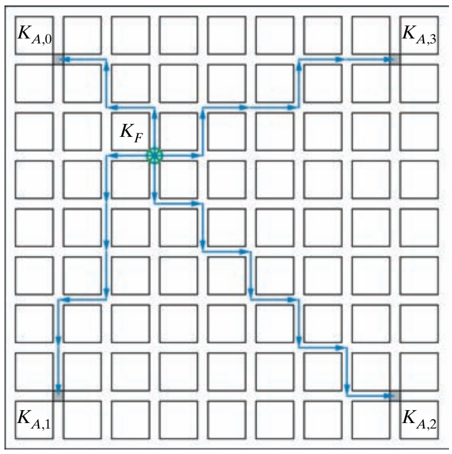

Fig. 12: Number of iteration steps until the support point is included: $d(K_F,K_{A,0}) = 4$, $d(K_F,K_{A,1}) = 7$, $d(K_F,K_{A,2}) = 10$, $d(K_F,K_{A,3}) = 7$

Further developmental potential of the method shown consists of, in the simplest case, the expansion of spatial structures under non-perpendicular-acting loads. As such, comparable behaviour in the interaction of subsystems is to be expected, with the occurrence of systemic behaviour similar to that shown here.

The challenge here is the static indeterminacy of the sub-systems (*Fig. 13*). The support forces F_0 and F_3 can no longer be determined solely with geometry and with the actions F_1 and F_2, but additional assumptions, for example, of material or joints are necessary.

Yet another challenge is the transmission of the systemic observation shown here onto similarly behaving spatial structures made up of discrete elements. These structures are of particular significance in the area of biomorphic architecture. Contemporary examples are the Olympic stadium in Beijing by Herzog and de Meuron and the Centre Pompidou in Metz by Shigeru Ban.

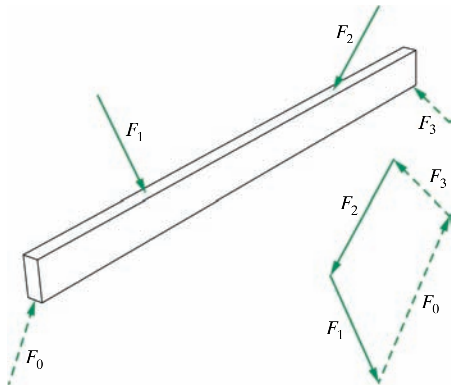

Fig. 13: Geometry and force diagram of a single bar in general case

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