On the extension of graphical statics into the 3rd dimension

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ABSTRACT: In the 19th century methods of graphical statics - systematized and published by Carl Culmann in 1866 – were used to study in more detail the relationship between form and force flow. Historically, they were superseded by analytical methods, due to the relative ease of numerical calculations and the increasing importance of experimentation the developing discipline of engineering with its need for precision. Nowadays, however, with the availability of digital tools and the possibility of visualization of geometry in 3D space there is a re-emerging interest within architecture in graphic statics. It is because of this that a visual dialogue about force flow based on graphic statics should be fostered because these methods offer a simple yet precise way to discuss the geometrical dependency of form and forces. The extended graphical statics (EGS) is part of an ongoing process of extension of the methods of graphic statics to the third dimension in order to overcome some of the limitations of Culmann's approach.

Similar to the situation in 2D it is possible to show, that by using the EGS in 3D any finite constellations of forces in general position can be reduced to a couple of forces, which represent all forces in intensity, direction and position. With that method the construction remains always spatially without any need for projections and always independent of any global coordinate system, necessary for numerical approaches. Like in 2D there are two linked diagrams, a form diagram and a force diagram, which mean that modifications in one diagram force an adjustment in the other diagram. With that method the dependency of form and forces in 3D can be seen without using projections and an active and synthetic designing process could begin. In previous research it has already been shown, that EGS allows considering the internal forces of a complex surface by using static determinate space trusses, which can be calculated graphical. These trusses can be summarized and three dimensional stress fields can be generated using CAD programs like e.g. Rhino in combination with Python and Mathematica.

In continuation of this research it is shown how EGS can be used to explore the 3D equilibrium by constructing a (trial) funicular polygon-net, which has to be seen as the 3D likeness of the (trail) funicular polygon in 2D. This netlike structure works as a closed system, so that every single cable of the net is determined. As in 2D, this funicular system helps to find out the 'neuralgic' points of a structure, so that with a few information the entire system can be considered. Using this method, important structural information of complex free forms can be examined that enables a geometrical and physical form finding process without using relaxation processes or FEM.

1 INTRODUCTION

The objective of a graphical approach to statics is to focus on geometric construction of the force flow in contrast to numerical calculation. That means the focus is on the logic of the inner force flow within a building structure rather than on the numeric evaluation of the building

structure itself. Graphic statics, therefore, fosters a process oriented and intuitive approach to force-oriented design.

Through the expansion of graphical methods of construction of force-flow into the third dimension, the foundation for a design methodology can be laid where architecture and engineering enter into an active and visual dialog. This is also the motivation for an ongoing research into a visual language for the flow of forces whose roots are to be found in the 'graphic statics' developed in the 19th century. ETH professor Karl Culmann, in particular, is considered one of the major advocates of this method and for him it was clear that 'drawing is the language of the engineer' (Mauerer 1998).

Architects primarily think and communicate using diagrams, sketches and plans that is drawing is the language of architects. The same is true for engineers. From this it is clear that graphic statics, as a method for the direct visual representation of the inner distribution of forces, is predestined to promote dialog between engineers and architects and can offer the foundation for fruitful discourse.

This is all the more valid since three-dimensional visualization is perfectly feasible using new CAD programs and so the method that was so far only used in 2D can intuitively be expanded to 3D. As a consequence, a direct relation between force and form can again be established. The first applications in this direction are already available for supporting frameworks subject to pure tensile and compression loading (Kilian 2005), and through them the experimental and elaborate design methods of Antonio Gaudi (Huerta 2006), Heinz Isler or Frei Otto become accessible for the drafting process. This creates an excellent basis for studying the inter-relation-ships between form and the progression of its inner forces more directly and thus more effectively and provides the opportunity to develop a design methodology based on an intuitive and easily comprehensible language. By developing the usability of the method by implementing into CAD programs a synthetic - instead of an analytic - design process can get started.

2 HISTORY

The central element of all drawing methods for representing the inner flow of forces is the investigation of the dependence of equilibrium and force polygons, as well as their representation in two geometrically dependent diagrams with different units. Such a description of equilibrium and force polygons is already found e.g. in Varignon's work 'Nouvelle Méchanique ou Statique' published in 1725 (Varignon 1725). For this research the most important roots are the projective correlation of the equilibrium and force polygon, which were mathematically investigated by Culmann as *'graphic statics'* (Culmann 1866) based on the *'newer geometry'* of Poncelet (Poncelet 1822) or Staudt's *'Geometrie der Lage'* published in 1847 (Staudt 1847). Because of the duality of the planar equilibrium and force polygon, Culmann called these diagrams reciprocal. Maxwell proved almost simultaneously (1864) *'that for non-central force systems two reciprocal figures arise if and only if one figure can be considered a projection of a polyhedron'* (Kurrer 2002). However, through the development of faster computing tools and the scientific processes (Westermann 2010) of the dawning 20th century, the graphical methods and their enormous potential as intuitive tools fell into oblivion, displaced more and more by analytical methods. Nowadays *'graphic statics'* plays only more of a subordinate role in engineering practice and has almost completely been displaced by analytical statics since the latter is more conducive to numerical analysis and therefore to calculations.

It must be recognized that all drawing methods known thus far for calculating the inner flow of forces operate in two-dimensional space. Even in the most recent research such as the *'Thrust Network Analysis'* of P. Block (Block 2009) investigations are carried out on projections, which is why the subject matter deals with planar, and are just valid for compression or tension only structures. If *'graphic statics'* is expanded into the third dimension, then both the representation of the equilibrium polygon (form diagram) and the setup of the force polygon (force diagram) must be done using spatial and thereby physical models or in digital space.

3 EXTENSION OF GRAPHICAL STATICS

3.1 *Number of resultants depending on the geometric position*

Planar force configurations can be consolidated into one equivalent force, of the resultant R, where an equivalent force is a force, which produces the same reaction forces as the sum of all individual forces using any static system. Different methods were developed for this purpose. The equilibrium polygon method deserves special mention in this connection. It allows the combination of a finite number of intersecting forces and a finite number of parallel and/or intersecting forces into one resultant in two-dimensional space. Intersecting or parallel forces can be consolidated into one resultant even in three-dimensional space by linking coplanar cases. On the other hand, it is not clear how skew forces can be combined graphically.

3.2 *Generic case: skew forces*

For a skew and thus generic force configuration in R^3 it should therefore be shown that it can be consolidated into a force pair (from many possible force pairs) through complete induction. This force pair is called (R_1 , R_2). What is true here as well is that an equivalent force pair using any static system brings about the same reaction forces as the sum of all individual forces. Without loss of generality, the following conditions must apply to three forces F_1 , F_2 , F_3 in general position:

1. All three forces $= 0$

2. All three forces by pair are not in a plane

3. In the form diagram:

$$\left[\left(\vec{a} + \vec{f_1}' \right) \times \vec{f_1}' + \left(\vec{c} + \vec{f_2}' \right) \times \vec{f_2}' + \left(\vec{e} + \vec{f_3}' \right) \times \vec{f_3}' \right]$$
$$- \left[\left(\vec{t} + \vec{r_1}' \right) \times \vec{r_1}' + \left(\vec{v} + \vec{r_2}' \right) \times \vec{r_2}' \right] = 0$$

$$\left[\left(\vec{t} + \vec{r_1}' \right) \times \vec{r_1}' + \left(\vec{v} + \vec{r_2}' \right) \times \vec{r_2}' \right] - \left[\left(\vec{t} + \vec{r_I}' \right) \times \vec{r_I}' + \left(\vec{v} + \vec{r_{II}}' \right) \times \vec{r_{II}}' \right] = 0$$

         possible pair of forces   additional, possible pair of forces

4. In the force diagram:

$$\left\| \vec{f_1}' + \vec{f_2}' + \vec{f_3}' \right\| - \left\| \vec{r_1}' + \vec{r_2}' \right\| = 0$$

$$\left\| \vec{r_1}' + \vec{r_2}' \right\| \quad - \quad \left\| \vec{r_I}' + \vec{r_{II}}' \right\| \quad = 0$$

         possible pair of forces   additional, possible pair of forces

For the initialization there are three forces F_1 , F_2 , and F_3 in whose action lines *g, h* and *i* are skewed against each other and whose direction and position are determined by two points each. *A* is freely chosen as the source of the coordinate system. Analogous to two-dimensional graphic statics, a three-dimensional force diagram is developed whose source is freely selectable. It is obvious, that a straight line *s* can be found, that intersects all three action lines *g, h* and *i*. So it is possible to generate two planes E_{jh} and E_{ni} which are intersecting in *s* (Figure 1).

By splitting F_2 and F_3 in two directions and F_1 in three directions it is possible to figure out the point *T* which is one point on the action of R_1. Because of the order of F_1 , F_2 , and F_3 in the force diagram, it is in the generic case not possible to split the whole force F_3 into the two

directions. That means that a remainder of the force F_3 is R_2.
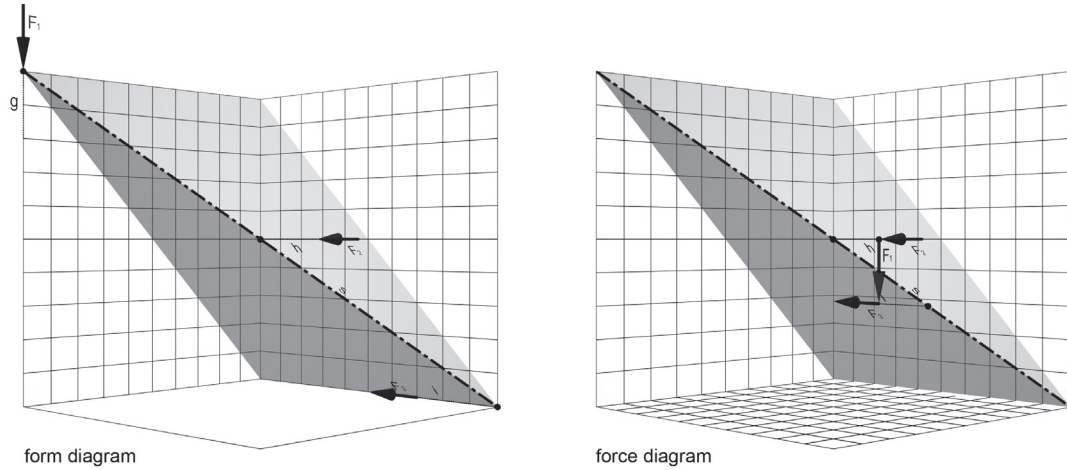

form diagram   force diagram

Figure 1. Intersecting line s

It is clear that the construction described creates only one of the infinite number of force pairs. Just like in the equilibrium polygon method it is completely irrelevant (cf. pitch) in which action line and at which point on the action line the construction is started but the geometric relationships of the form diagram must be transferred to the force diagram. In this respect the construction can be executed so that the remaining force lies on the action line of F_1, F_2 or F_3 (Figure 2).

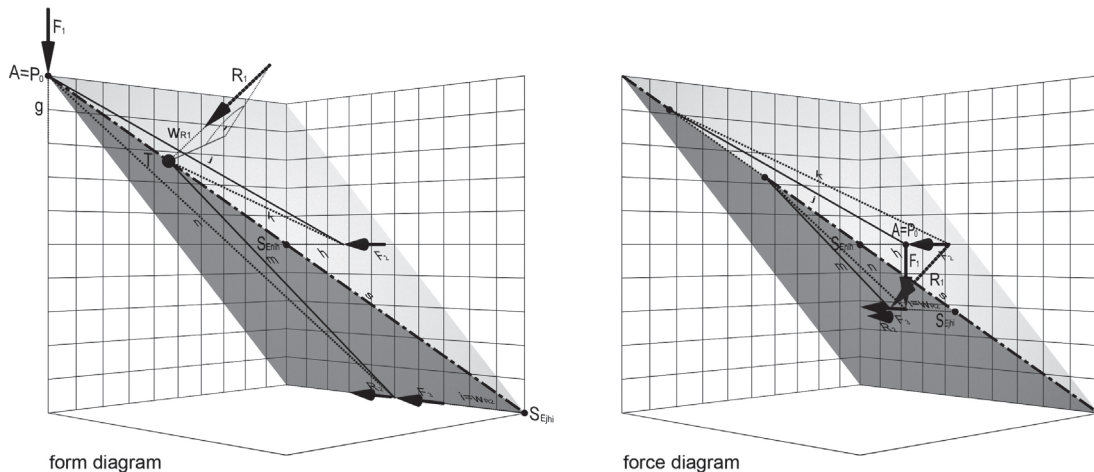form diagram   force diagram

Figure 2. Three skew forces subdivided into R_1 and R_2

In an iterative process three skew forces combined into one force pair can be linked to another force and so a finite number of skew forces can be reduced to one force pair through complete induction. This force pair corresponds in intensity and direction to the sum of all forces.

$$(F_1 + F_2 + F_3 + F_4 + \cdots + F_n) - (R_{1,} + R_2) = 0$$

and

$$(R_1, R_2) \qquad\qquad - \qquad (R_I, R_{II}) \qquad\qquad = 0$$

   possible pair of forces   additional, possible pair of forces

4 SPATIAL FUNICULAR POLYGON NET

4.1 *Most simple spatial net using R_1 and R_2*

As shown in chapter 3, any configuration of skew forces can be consolidated into two equivalent forces. This pair of forces - using any statically determined system - brings about the same reaction forces as the sum of all individual forces. So it is clear, that working with these two forces helps to find out the 'neuralgic' points of a structure, so that out of these information the entire system can be considered. On the other hand it means, that the simplest net to get an overview is out of these two forces connected with any three supports.

4.2 *Extension of the funicular line (2D) to a spatial polygon net (3D)*

All the research is about showing, that planar graphic statics is a special case of the 3D. Therefore, it is evident to discover the analog system of the 2D funicular system in 3D.

In 2D the difference by the force diagram between using the reslutant or the origin forces is, that the forcepolygon gets refined (Figure 3).

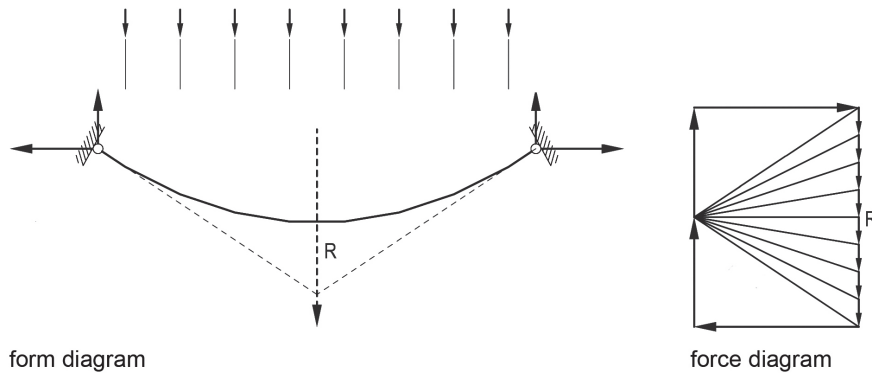form diagram   force diagram

Figure 3. Refinement by using polygonal approximation

In planar graphic statics there are many conventions, so that the order of the forcepolygons defines the shape of the construction or vice versa. However from a numeric point of view in 2D e.g. by two forces 4 possible constellations of (force-)polygonlines with a freely selected sag are existing. In 3D by two forces 12 possible constellations of polygonlines with a freely selected sag are existing (Table 1). That shows that with increasing the number of forces the number of possible polygonlines increases too. That means that for a practical implementation it is necessary to define conventions.

Table 1. Comparison of the degrees of freedom

Number of possibilities for	2D	3D
starting with the polygon at any support	2	3
connecting the first line of action	2	2
connecting the second line of action	1	1
connecting the second support	1	2
connecting the third support	-	1
Total	4	12

In the two-dimensional force diagram the line of action of the forces are always connected with one single line/forcevector. In 3D two lines/forcevectors – remember the plane – are necessary to connect the origin forces. This leads to deflections, which are – depending on the origin forces – visible as compactions or 'vortexisations'. In planar graphic statics just the conventions prevent overlapping forces, the 'vortexisation' is unique in 3D.

All this polygonlines whether in 2D or in 3D are unstable and just valid for a singular force combination. For getting a system that resists any force combination in 2D a net - that means a statically determined truss - has to be constructed. In 3D the eqivalent would be a spatial truss out of tetrahedrons.

This raises the question of the role of a triangulated net in 3D. Is there a possibility to use a triangulated net as a discretized version e.g. of a shell structure to explore the force flow? The problem is quite similar to statically indeterminated trusses. For solving it in a graphic way it is necessary to choose some forces. For providing a usefull assumption there has to be an existing idea of the final form and so it is no more a 'design-free' descripiton of the force flow.

4.3 *How to create a net using R1 and R2*

First it is necessary to define any three supports instead of two. It is obvious that for a statically determined system in space one support has to force reactions in one direction, the second in two directions and the third has to be fixed. Now the support forces out of R_1 and R_2 can be determined using superposition. For that the simplest way is to work with tetrahedrons. For R_1 as well as for R_2 a tetrahedron between any point on the action line – except the intersecting point on the connection plane between the three supports – has to be build. This tetrahedron connects on the one hand the action line of the force with the three supports, as well as the three supports with each other. By solving graphically the support forces out of R_1 and R_2 and a subsequent superposition the support forces are determined. Out of this information it is possible to fix the closing plane – instead of a closing string in 2D - in the force diagram (Figure 4).
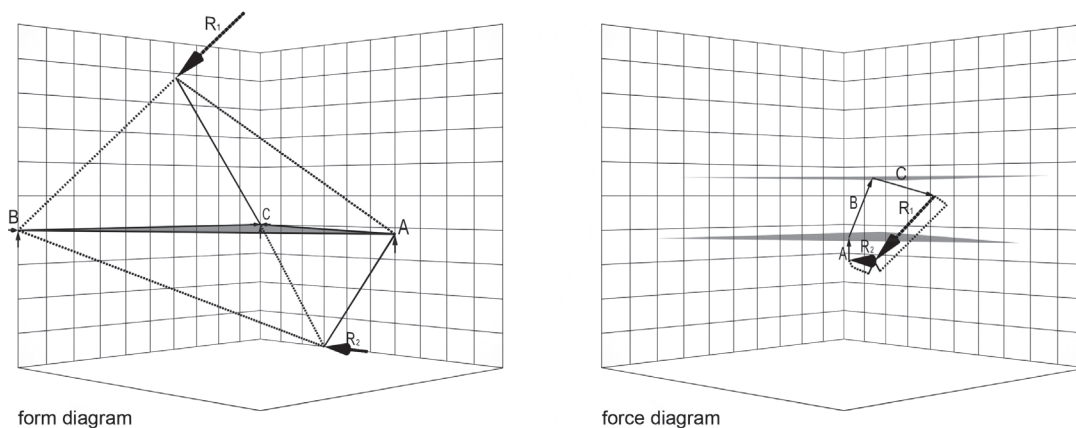

Figure 4. Tetrahedron construction for fixing the position of the closing plane in the force diagram

That offers the possibility to construct a polygon line using connecting planes. The outer forces are the same. In 3D it is necessary to insert planes – instead of lines – for getting a connection between the origin force vectors.

The sag can be freely selected – like in 2D – except being zero and of course the pole can be shifted along the closing plane in the force diagram or along the line of action of the forces in the form diagram (Figure 5).

It is obvious, that the inner forces are changing from compression to tension or vice versa by shifting the sag beyond the closing plane. This dependency shows very nice, that the graphical statics in 2D are just a special case of the 3D what can be e.g. shown by the closing string. The closing string (2D) gets a closing plane (3D) as pending. It has to be mentioned, that all action

lines are not coplanar, so they are crossing – not intersecting – each other in space in relation to the normal of the closing plane. This phenomenon leads into a 'vortexisation' of the polygon line in the form diagram depending on the selected sag.
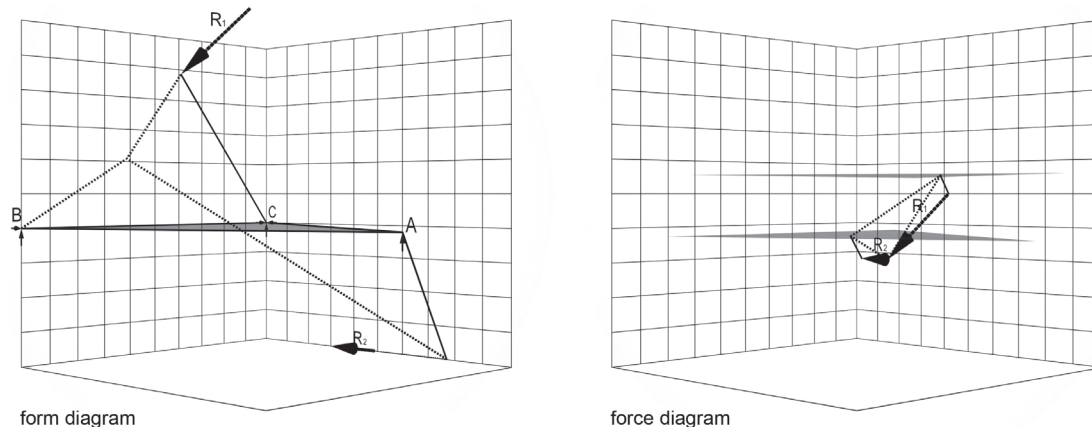

Figure 5. Polygon line net with a freely selected sag

5 USING THE EXTENDED GRAPHICAL STATICS AS A DESIGN TOOL

5.1 *Fundamental problem*

Contemporary CAD programs like e.g. Rhino® in combination with Grasshopper® have changed fundamentally design processes. Complex shapes, which made in former times an object scarcely graspable can now be easily generated and produced by digital tools. It appears that any geometrical challenge can be solved. This new freedom of design has often the unfortunate effect of leading against the laws of physics - that sometimes ends in unsatisfactory results.

The challenge is to make the complex inner force flow of a free-form object easy understandable by breaking down to the essential information. FEM methods can only make a limited contribution to this problem, because they are just analytical tools. By implementing the extended graphical statics into the CAD environment, a synthetic tool is provided, that is entirely based on geometry.

5.2 *Integration of extended graphical statics into Rhino®*

The CAD program Rhino® in combination with the plug-in Grasshopper® offers the possibility to combine simple elements for controlling parameters and input data like e.g. sliders or 'get-point command' with python® scripting elements. This ensures an intuitive and very easy access for the 'generic' user. It is very intuitive to change e.g. a point in the form diagram and immediately the adjustment in the force diagram follows. This dual consideration helps to make design decisions on a multi layered level.

5.3 *First designs made by using the application*

In the second year course of structural design of Department of Architecture of the Swiss Federal Institute of Technology Zurich (ETH) the 3D graphical statics tool have been applied. Students analyzed the 3D equilibrium by using a simple combination out of tetrahedrons. This space frame is a determinate system and the inner force flow can be easy understood with the spatial force diagram. The insightful process began by using the linked diagrams to get an immediate feedback when the geometrical form was changed. By studying the form and controlling the inner forces, it was possible to figure out a specific form, what was neither an optimization of the intensity of the forces nor a pure result of a design idea.

In a first step students have calculated the support forces by the described method of superposition (4.3). In a second step a rough outline was framed by choosing a sag between the support plane and the position of random points on the action lines of the resultants. Step by step the form was traced with tetrahedrons and the inner forces were calculated graphically. In the end rather than optimization the goal was to balance the freedom of design with the laws of physics, which result you can e.g. see at Figure 6.
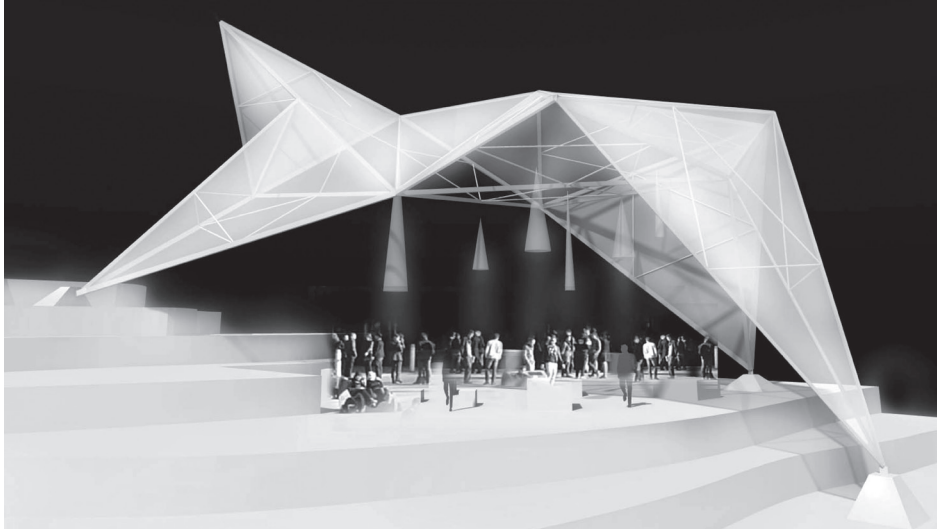

Figure 6. Student project using 3D graphical statics at Department of Architecture, ETH Zurich

6 CONCLUSION

The extended graphical statics has the potential to get a very powerful tool for understanding the inner force flow, especially in the field of free-form design. For that it is necessary to optimize the implementation in the different digital tools. The great thing about it is, that the discussions by involved persons can be carried out in the common language of drawing. It is very important to mention, that the method does not present an overly simplified view of interrelationships. It is mathematically precise and meets the standards of engineering sciences. This method of structural analysis fosters a holistic understanding of the interaction of form and structure, in other words the understanding of the internal forces within a built structure that help determine its form.

By developing this method there will be an alternative to the common FEM programs. Of course the extended graphical statics will not be the sole 'miracle subsidy cure' to solve existing problems, but it will be the logical evolution of the 'graphical statics' of Culmann, which has been the theoretical principle of many outstanding buildings like e.g. the 'Tour Eiffel' in Paris.

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