R. Stouffs, P. Janssen, S. Roudavski, B. Tunçer (eds.), *Open Systems: Proceedings of the 18th International Conference on Computer-Aided Architectural Design Research in Asia (CAADRIA 2013)*, 843–852. © 2013, The Association for Computer-Aided Architectural Design Research in Asia (CAADRIA), Hong Kong, and Center for Advanced Studies in Architecture (CASA), Department of Architecture-NUS, Singapore.

STATICALLY MOTIVATED FORMFINDING BASED ON EXTENDED GRAPHICAL STATICS (EGS)

Maximilian J. SCHREMS and Toni KOTNIK
Swiss Federal Institute of Technology Zurich, Zurich, Switzerland
schrems@arch.ethz.ch, kotnik@arch.ethz.ch

Abstract. In the 19th century it was common to use graphical methods for study the relationship between form and force. The central element of all drawing methods for representing the inner flow of forces was the investigation of the dependence of equilibrium and force polygons, as well as their representation in two geometrically dependent diagrams with different units. This research is part of an on going project to extend the methods of 'graphical statics' of Carl Culmann (1866) to the third dimension in order to overcome some of the 2D-limitations of this approach. It is focused on the construction of resulting force within 3D and the utilization within discrete space frames (tetrahedrons) in equilibrium. The objective of the EGS is to focus on constructing in contrast to calculating. That means that the logic of the inner force flow leads to a process-oriented and visible approach of design, which gets computationally accessible. With the use of digital tools and increasing importance of performative methods of form-finding a renewed interest in these vector-based geometric methods of construction of force flow has occurred. This maybe will give the possibility to get an alternative to the common form finding methods by relaxation processes and analysis by FEM.

Keywords. Graphical statics; 3D equilibrium; form finding method.

1. Introduction

Through the expansion of graphical methods of construction of force-flow into the third dimension, the foundation for a design methodology can be laid where architecture and engineering enter into an active and visual dialog. This is also the motivation for an on going research into a visual language for the flow of forces whose roots are to be found in the 'graphic statics' developed in the 19th century by Karl Culmann (Mauerer, 1998).

Because architects primarily think and communicate by using diagrams, sketches and plans it is obvious, that drawing is the language of architects. The same is true for engineers. From this it is clear that 'graphic statics' (Culmann, 1866), as a method for the direct visual representation of the inner distribution of forces, is predestined to promote dialog between engineers and architects. This approach will help to bring the linked professions together (Shea and Cagan, 1999) and not to emphasise the different point of view.

This is all the more valid since three-dimensional visualization is perfectly feasible using new CAD programs and so the method that was so far only used in 2D can intuitively be expanded to 3D. The first applications in this direction are already available for supporting frameworks subject to pure tensile and compression loading (Kilian and Ochsendorf, 2005), and through them the experimental and elaborate design methods of Antonio Gaudi (Huerta, 2006), Heinz Isler or Frei Otto become accessible for the drafting process. By developing the usability of the method by implementing into CAD programs a synthetic – instead of an analytic – design process can get started.

The EGS has the potential to get a very powerful tool for understanding the inner force flow, especially in the field of free-form design. It is very important to mention, that the method does not present an overly simplified view of interrelationships. It is mathematically precise and meets the standards of engineering sciences. This method of structural analysis fosters a holistic understanding of the interaction of form and structure, in other words the understanding of the internal forces within a built structure that help determine its form.

2. History

The central element of all drawing methods for representing the inner flow of forces is the investigation of the dependence of equilibrium and force polygons, as well as their representation in two geometrically dependent diagrams with different units. Such a description of equilibrium and force polygons is already found e.g. in Varignon's (1725) work 'Nouvelle Méchanique ou Statique' published in 1725. For this research the most important roots are the projective correlation of the equilibrium and force polygon, which were mathematically investigated by Culmann (1866) as 'graphic statics' based on the 'newer geometry' of Poncelet (1822) or Staudt's (1847) 'Geometrie der Lage' published in 1847. Maxwell (1864) proved almost simultaneously 'that for non-central force systems two reciprocal figures arise if and only if one figure can be considered a projection of a polyhedron' (Kurrer, 2002). However, through the development of faster computing tools and the scientific processes (Westermann, 2010) of the dawning 20th century, the graphical methods and their enormous potential as intuitive tools fell into oblivion, displaced more and more by analytical methods. Nowadays 'graphic statics' plays only more of a subordinate role in engineering practice and has almost completely been displaced by analytical statics since the latter is more conducive to numerical analysis and therefore to calculations.

It must be recognized that all drawing methods known thus far for calculating the inner flow of forces operate in two-dimensional space. Even in the most recent research such as the 'Thrust Network Analysis' of Block (2009) investigations are carried out on a projection, which is why the subject matter deals with planar, and are just valid for compression or tension only structures.

3. Extension of Graphical Statics

3.1. NUMBER OF RESULTANTS

Planar force configurations can be consolidated into one equivalent force, of the resultant R, where an equivalent force is a force, which produces the same reaction forces as the sum of all individual forces using any static system. Different methods were developed for this purpose. The equilibrium polygon method deserves special mention in this connection. It allows the combination of a finite number of intersecting forces and a finite number of parallel and/or intersecting forces into one resultant in two-dimensional space. Intersecting or parallel forces can be consolidated into one resultant even in three-dimensional space by linking coplanar cases. On the other hand, it is not clear how skew forces can be combined graphically.

3.2. GENERIC CASE: SKEW FORCES

For a skew and thus generic force configuration in R^3 it should therefore be shown that it can be consolidated into a force pair (from many possible force pairs) through complete induction. This force pair is called (R_1, R_2). What is true here as well is that an equivalent force pair using any static system brings about the same reaction forces as the sum of all individual forces. Without loss of generality, the following conditions must apply to three forces F_1, F_2, F_3 in general position:

1. All three forces = 0
2. All three forces by pair are not in a plane
3. In the form diagram:

$$\left[\left(\vec{a}+\overrightarrow{f_1}'\right)\times\overrightarrow{f_1}'+\left(\vec{c}+\overrightarrow{f_2}'\right)\times\overrightarrow{f_2}'+\left(\vec{e}+\overrightarrow{f_3}'\right)\times\overrightarrow{f_3}'\right]-$$
$$\left[\left(\vec{t}+\overrightarrow{r_1}'\right)\times\overrightarrow{r_1}'+\left(\vec{v}+\overrightarrow{r_2}'\right)\times\overrightarrow{r_2}'\right]=0$$

$$\left[\left(\vec{t}+\overrightarrow{r_1}'\right)\times\overrightarrow{r_1}'+\left(\vec{v}+\overrightarrow{r_2}'\right)\times\overrightarrow{r_2}'\right]-\left[\left(\vec{t}+\overrightarrow{r_I}'\right)\times\overrightarrow{r_I}'+\left(\vec{v}+\overrightarrow{r_{II}}'\right)\times\overrightarrow{r_{II}}'\right]=0$$

possible pair of forces additional, possible pair of forces

4. In the force diagram:

$$\|\overrightarrow{f_1}'+\overrightarrow{f_2}'+\overrightarrow{f_3}'\|-\|\overrightarrow{r_1}'+\overrightarrow{r_2}'=0\|$$

$$\|\overrightarrow{r_1}'+\overrightarrow{r_2}'\| \quad - \quad \|\overrightarrow{r_I}'+\overrightarrow{r_{II}}'\|=0$$

possible pair of forces additional, possible pair of forces

For the initialization there are three forces F_1, F_2, and F_3 in whose action lines *g, h* and *i* are skewed against each other and whose direction and position are determined by two points each. *A* is freely chosen as the source of the coordinate system. Analogous to two-dimensional graphic statics, a three-dimensional force diagram is developed whose source is freely selectable. It is obvious, that a straight line *s* can be found, that intersects all three action lines *g, h* and *i*. So it is possible to generate two planes E_{jh} and E_{ni} which are intersecting in *s* (Figure 1).

By splitting F_2 and F_3 in two directions and F_1 in three directions it is possible to figure out the point *T* which is one point on the action of R_1. Because of the order of F_1, F_2, and F_3 in the force diagram, it is in the generic case not possible to split the whole force F_3 into the two directions. That means that a remainder of the force F_3 is R_2.

It is clear that the construction described creates only one of the infinite number of force pairs. Just like in the equilibrium polygon method it is completely

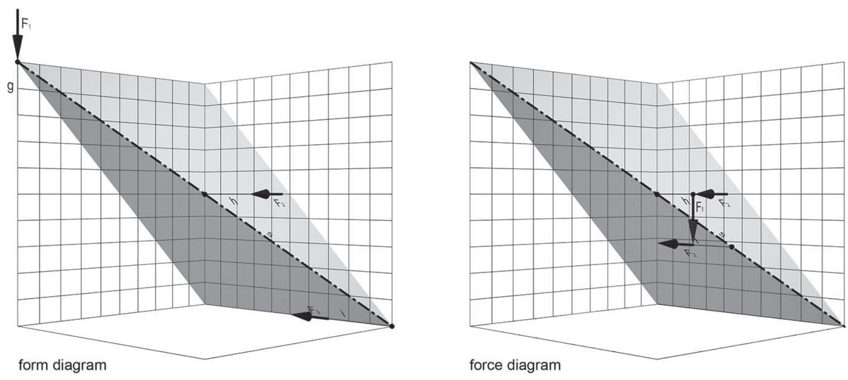

Figure 1. Intersecting lines.

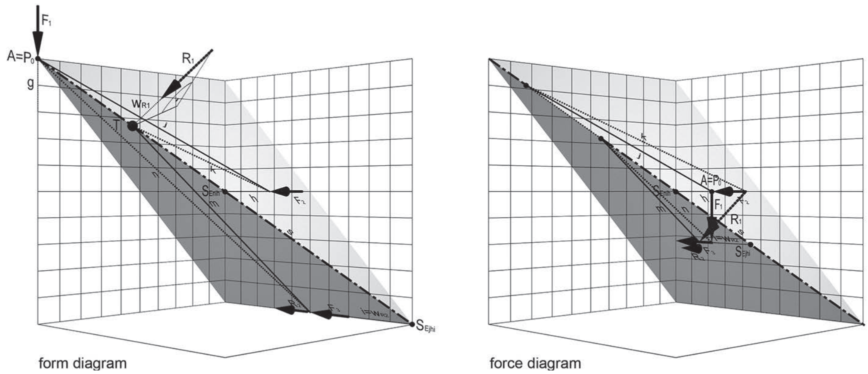

Figure 2. Three skew forces subdivided into R_1 *and* R_2*.*

irrelevant (cf. pitch) in which action line and at which point on the action line the construction is started but the geometric relationships of the form diagram must be transferred to the force diagram. In this respect the construction can be executed so that the remaining force lies on the action line of F_1, F_2 or F_3 (Figure 2).

In an iterative process three skew forces combined into one force pair can be linked to another force and so a finite number of skew forces can be reduced to one force pair through complete induction. This force pair corresponds in intensity and direction to the sum of all forces.

$$(F_1 + F_2 + F_3 + F_4 + \ldots + F_n) - (R_{1,} + R_2) = 0$$

and

$$\underset{\text{possible pair of forces}}{(R_1, R_2)} \quad - \quad \underset{\text{additional, possible pair of forces}}{(R_I, R_{II}) = 0}$$

4. Digital Approach

The extension of graphical statics will only be fully effective if the interrelationships between form and force will be visually comprehensible even in 3D and facilitates analysis as well as an active and synthetic design process.

4.1. SELECTION OF THE CAD ENVIRONMENT

Rhinocerus 3d® offers the possibility to act very intuitive and visually in an interactive 3D environment. For many designers and architects it is common to use this digital tool. The same applies to Grasshopper® that allows generative modelling by implement parametric elements without having a deep knowledge of programming and scripting. Because Rhinocerus 3d® and Grasshopper® are common tools in the architectural world they were chosen. It is obvious, that the standard functions of Grasshopper® would lead to a complex control diagram. However, that's why it is appropriate to implement a small Python® script. It is clear, that this implementation makes it not longer to a pure geometric calculation but rather to an algebraic method, but the user logic is still geometric and that allows a geometric and out of it synthetic approach.

As shown in section 3, any configuration of skew forces can be consolidated into two equivalent forces. This pair of forces – using any statically determined system – brings about the same reaction forces as the sum of all individual forces. The challenge is to find a simple way to start the described iterative process, especially because in every step of iteration the degrees of freedom have to be eliminated by decisions of the user. It is clear, that it would be possible to use a randomizer for doing this as well. But this depends on the use of the result.

4.2. INITIALIZATION

The first step is to initialize the system in the form diagram. To get the straight-line equation of the lines of action of the different forces it is necessary to know two points on each line of action. Out of this the relevant unit vector can be determined (Figure 3).

In the force diagram the unit vectors must be multiplied scalarly. This can be achieved easily using a slider in Grasshopper® (Figure 4). For beginning a random starting point *ak* has to be set.

Analogous to two-dimensional graphic statics, a three-dimensional force diagram is developed whose source is freely selectable. After A has been set as the source in the form diagram, it also makes sense to define *ak* = P'_0. The order of

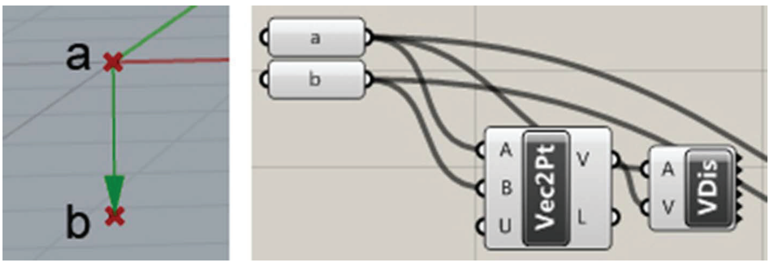

Figure 3. Initialization by using 'getPoint-' and 'Vec2Pt-function'.

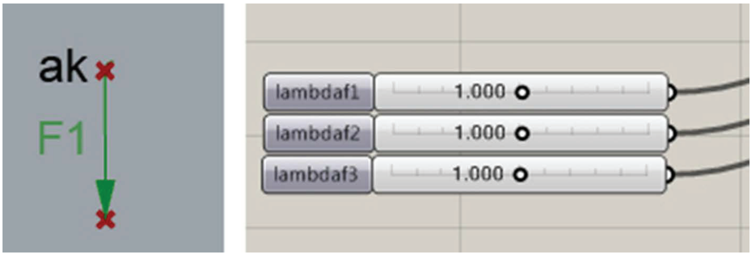

Figure 4. Scaling by using a 'slider'.

force vectors f_1', f_2' and f_3' can be any but one must make allowances for the order in the following construction.

4.3. IMPLEMENTATION

All the steps of the construction described in 3 can be easy and in time calculated by a small Python® script. The construction creates only one of the infinite numbers of force pairs. Just like in the equilibrium polygon method it is completely irrelevant in which action line and at which point on the action line the construction is started but the geometric relationships of the form diagram must be transferred to the force diagram (Figure 5).

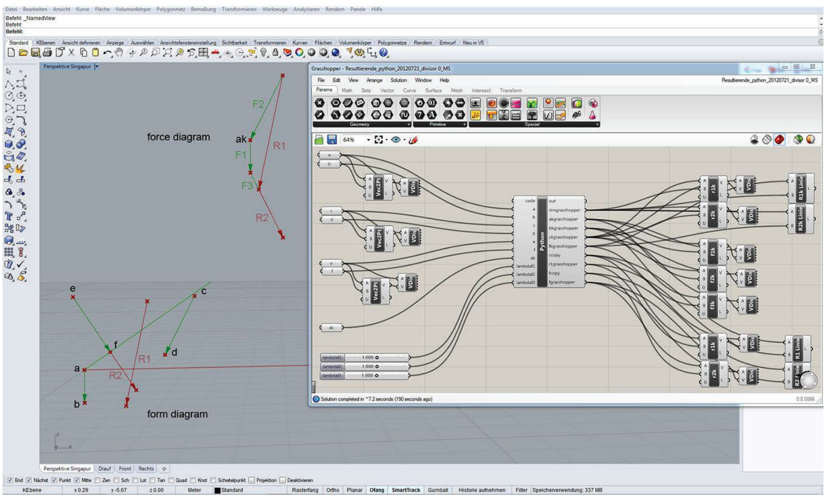

Figure 5. Full control panel with three forces.

4.4. COMPLETE INDUCTION

In an iterative process three skew forces combined into one force pair can be linked to another force and so a finite number of skew forces can be reduced to one force pair through complete induction. This force pair corresponds in intensity and direction to the sum of all forces

4.5. ADVANTAGES OF PARAMETRIZATION

The interactive and linked diagrams are offering – in combination with the Grasshopper®-Python® surface – the possibility to adjust the resultant in any direction by changing either the position of the lines of action or by an adaptation of the intensity of the forces. The adaption can be taken so far, that the constellation and intensity of the forces can be consolidated into one force instead of a force pair. In many chases wee geometrical operations simplify the force flow enormous.

4.6. SUPPORT FORCES CALCULATED BY SUPERPOSITION

First it is necessary to define any three supports. For a statically determined system in space one support has to force reactions in one direction, the second in two directions and the third has to be fixed. Now the support forces out of and can be determined using superposition. For that the simplest way is to work with tetrahedrons. For as well as for a tetrahedron between any point on the action line – except the intersecting point on the plane between the supports – has to be build. By solving graphically the support forces out of and and a subsequent superposition the support forces are determined. Out of this information it is possible to fix the closing plane – instead of a closing string in 2D – in the force diagram (Figure 6).

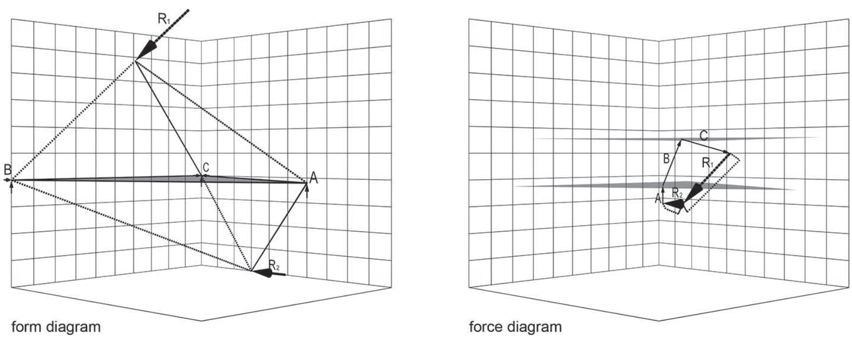

Figure 6. Tetrahedron construction for calculating the support forces.

5. Discussion

5.1. COMPARISON WITH EXISTING METHODS

Nowadays it is common to use relaxation methods for form finding and FEM for analysis. It is obvious, that both methods have their strengths and weaknesses. FEM is a sophisticated tool to allow analysis of an already designed form, but gives no direct indication how to adapt the form to get a better inner force flow. On the other hand relaxation processes are providing the possibility to optimize a rough form but just in a way of optimization. Both is only partially helpful for designing a specific form.

Because EGS is purely geometric and free of auxiliary quantities such as moments – which are used by analytic methods – it allows a resultant consideration based on geometric dependencies (e.g. diversions) and thus shows the relationship of force and form directly. When shifting forces in 3D in analytical-numerical statics, the consideration of the force pair, consisting of the resultant force and the resultant moment, is the significant component in the analysis. This type of consideration, always requires a shifting of the resultant force into a freely selectable reference point, which by definition determines the resultant moment. However, since a skew force pair arises in the generic case, the reference point is often chosen so that a force skew and thus a colinear position of the resultant force vector and of the resultant moment vector are set. Basically 'every force system can be reduced in an infinite number of ways to two forces skewed against each other, the so-called force cross' (Szabo, 1975). This is precisely what is being done in the EGS construction in just with the geometric dependency with the difference, that the result can be visually traced.

As EGS is free of material behaviours and just based on geometrical dependency, it helps to practice form finding without numerical detours. This will help to design free form, lightweight or kinematic structures in a synthetic instead of 'trail and error' process.

5.2. EXPERMINENTAL DESIGNS AT ETH ZURICH

In the 2nd year course of structural design of Department of Architecture the EGS tool have been applied. Students analyzed the 3D equilibrium by using a simple combination out of tetrahedrons. This space frame is a determinate system and the inner force flow can be easy understood with the spatial force diagram. By studying the form and controlling the inner forces, it was possible to figure out a specific form, what was neither an optimization of the forces nor a pure result of a design idea (Figure 7).

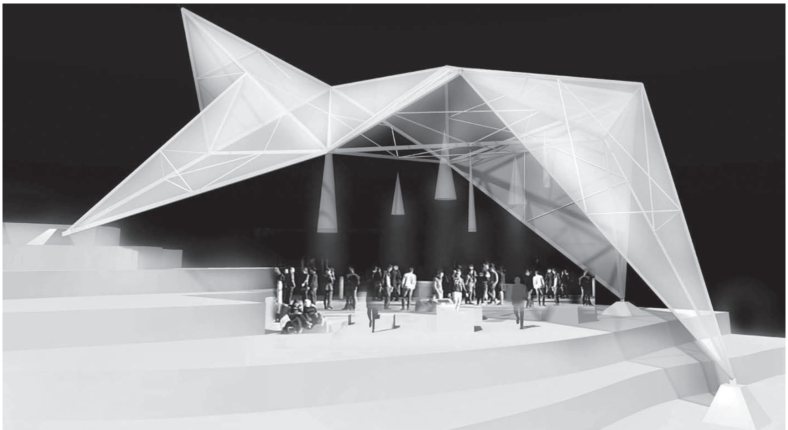

Figure 7. Student project using EGS.

6. Conclusion

By developing this method there will be an alternative to the common FEM programs. Of course the EGS will not be the sole 'miracle subsidy cure' to solve existing problems, but it will be the logical evolution of the 'graphical statics', which has been the theoretical principle of outstanding buildings.

References

Block, P.: 2003, *Thrust Network Analysis – Exploring Three-dimensional Equilibirium*, Princeton University, Princeton, NJ.
Culmann, K.: 1866, *Die graphische Statik*, Meyer und Zeller, Zürich.
Huerta, S.: 2006, Structural design in the work of Gaudi, *Architectural Science Review*, **49**(4), 324–339.
Kilian, A. and Ochsendorf, J.: 2005, Particle-spring systems for structural form finding, *IASS Journal*, **46**(2), 77–84.
Kurrer, K.-E.: 2002, *Geschichte der Baustatik*, Ernst und Sohn, Berlin.
Mauerer, B.: 1998, *Karl Culmann und die graphische Statik*, Verlag für Geschichte in Wissenschaft und Technik, Berlin.
Poncelet, J.-V.: 1822, *Traité des propriétés projectives des figures*, Paris.
Shea, K. and Cagan, J.: The design of novel roof trusses with shape annealing: assessing the ability of a computational method in aiding structural designers with varying design intent, *Design Studies*, **20**(1), 3–23.
Staudt, K.: 1847, *Geometrie der Lage*. Nürnberg.
Szabo, I.: 1975, *Einführung in die technische Mechanik*, Springer, Berlin.
Varignon, P.: 1725, *Nouvelle méchanique ou statique*, C. Jombert, Paris.
Westermann, A.: 2010, "Mathematik zwischen Leitdisziplin und Hilfswissenschaften". Available from: <http://www.ethistory.ethz.ch/besichtigungen/touren/vitrinen/studieren/vitrine23/> (accessed 20 November 2012).