Strut and Tie Networks – An Approach to Numerical Curved Stress Fields

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Summary
This paper presents a new method for form finding of general strut and tie networks with no limitations to the direction of loads and the arrangement of nodes and members. Based on this generally applicable approach Curved Stress Fields are approximated, which are able to vividly describe internal forces of solid structural surfaces. Yield Stress Fields define bounds of the eccentricity of Curved Stress Fields towards the centre surface of a structure and thus allow a consideration of structural resistance in the form-finding process.

Keywords: *structural surfaces made of reinforced concrete, form-finding, equilibrium solutions, lower-bound analysis, Yield Stress Fields*

1. Introduction

Techniques applied in statically motivated form finding of structural surfaces are changing from experimental to digital methods, as the effort made for the generation of digital models is considerably smaller. Irrespective of advances, which have been made in the field; research has been mainly focussed on statically optimal forms. However, in architecture the use of curved surfaces has dramatically changed over the last decades: from being statically dominated to an increasingly architecturally determined formal language. To consider both aspects, the architectural freedom of design and the engineer's pursuit of efficient structures, the proposed Curved Stress Field method allows conscious deviations from the statically ideal form within the bounds of structural resistance.

The design of reinforced concrete structures by stress fields has been a major inspiration for the proposed method of Curved Stress Fields. It was first sketched by Drucker [1] and much later refined to a practically applicable method by Muttoni et al. [2]. Compared to strut and tie models it allows a more detailed analysis of the behaviour of reinforced concrete structures subjected to in-plane stresses, while achieving the same vividness. The method is based on the lower bound theorem of the theory of plasticity and a rigid-plastic material model.

A Curved Stress Field is metaphorically speaking an infinitely tight network of thrust lines and funicular curves, which form a continuous, curved biaxial stress state. The generation of continuous Curved Stress Fields is possible [3, 4], but lacks practical applicability due to the mathematical problems connected to the solving of partial differential equations. Thus, to enable an intuitive and efficiently applicable tool a numerical approximation of Curved Stress Fields through curved strut and tie networks is proposed.

2. Three-Dimensional Equilibrium of Strut and Tie Systems

Thrust Network Analysis [5], which is a method for the generation of discrete compression-only systems, allows an intuitive control of the form finding process. It is based on a projective relation between a planar initial system and its equilibrated counterpart and uses Maxwell's duality between the geometry of the initial network and its internal forces to intuitively control the member forces of the initial system. Thrust Network Analysis aims at form finding and analysis of vaults. A solution, which is compression-only and lies in between the defined intrados and extrados of a vault, is determined using a linear optimisation process.

While vaults made of stones or masonry are limited to compression-only forms, reinforced concrete shells allow also tensile stresses. Besides concrete shells under pure compression, for many reinforced concrete shells a combination of compressive and tensile stresses is used to gain structural stability and great slenderness. Thus, a method, which is intended to be used for reinforced concrete structures, must include the possibility of considering tensile forces.

The proposed approach for the determination of the form of a curved strut and tie network is a transformation of a kinematic initial system into an equilibrium state for a particular load case. The set up of the initial system defines the arrangement of nodes and members, while neither node positions nor lengths of members are fixed to their initial value. External loads and supports are assigned to the nodes of the initial system.

2.1 Initial and Equilibrated System

Initial member forces are assumed to be acting along members of the initial system. Within the scope of the proposed method an initial system does not have to meet equilibrium conditions, neither considering the applied external loads nor solely initial member forces. Thus, the magnitude of initial member forces is in general arbitrary. Exceptions to this supposition form initial systems, which are assumed to be a projection of the equilibrated system. The proposed definition of the relation between member forces of the initial and the equilibrated system illustrated in Fig. 1 forms the basis of the generation of the form of equilibrated strut and tie systems.

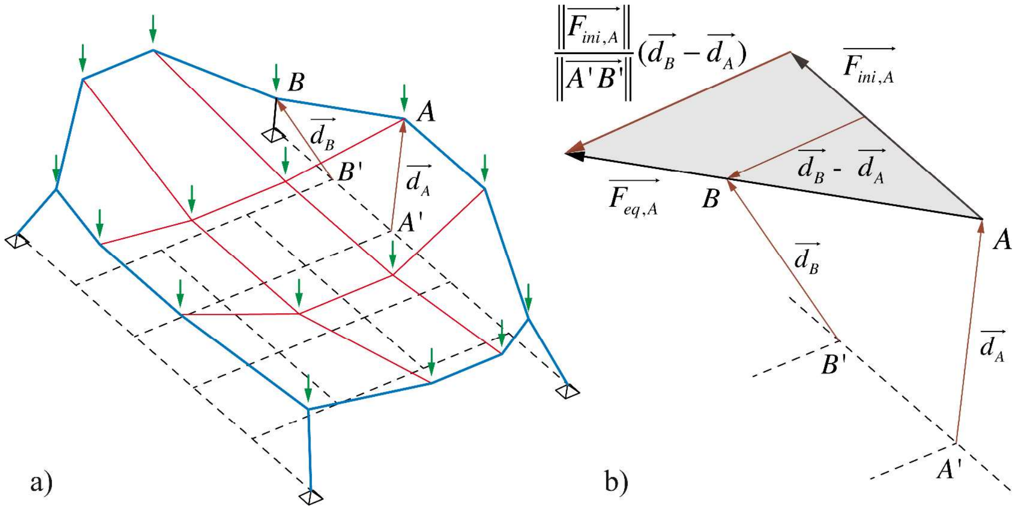

Fig. 1: Definition of the relation between member forces of the initial (dashed lines) and the equilibrated system (solid lines); a) exemplary initial and equilibrated system (blue: compression members, red: tension members); b) relation between member forces of the initial and the equilibrated system

The definition of the relation of member forces supposes that the ratio of their magnitudes is proportional to the ratio of lengths of the members. Equation (1) results from this basic assumption.

$$\overrightarrow{F_{eq,A}} = \frac{\left\|\overrightarrow{F_{ini,A}}\right\|}{\left\|\overrightarrow{A'B'}\right\|}\left(\overrightarrow{A'B'} + \overrightarrow{d_B} - \overrightarrow{d_A}\right) \qquad (1)$$

2.2 System of non-linear equations

At each node of the initial system an equilibrium condition can be formulated. The nodal equilibrium condition defines that the sum of all member forces and the applied external load must result in zero. Equation (2) shows the equilibrium condition for an exemplary node A.

$$\sum \overrightarrow{F_{eq,A}} + \overrightarrow{Q_A} = \vec{0} \qquad (2)$$

At an exemplary supported node C a vector of reaction forces is added to the equilibrium condition (3).

$$\sum \overrightarrow{F_{eq,C}} + \overrightarrow{Q_C} + \overrightarrow{R_C} = \vec{0} \qquad (3)$$

The equilibrium conditions of all nodes of the initial system form a system of non-linear equations, when equation (1) is substituted for the vectors of the member force of the equilibrated system in equation (2) and (3).

2.3 Control over the generation of strut and tie systems

The parameters of the system of equations describe magnitudes of initial member forces, components of displacement vectors of nodes and components of reaction forces. When using the proposed approach the scalar parameters outnumber the scalar equations. Control over the generation process of an equilibrated strut and tie system is gained by assigning values to these supernumerary parameters. The number of supernumerary parameters coincides with the sum of the number of members and the triple of the number of supported nodes. The set of parameters, which is considered to be supernumerary, is not predetermined. Hence, besides the actual assignment of values the selection of a particular set of supernumerary parameters is also decisive for the result of the generation process.

Components of displacement vectors of nodes can be directly set. The assignment of a value to one of the components results in limiting the further displacement of a node to a plane. In case of two predetermined components a direction of displacement is defined. The same effects are gained by defining relations between the components of displacement of a node, but allow limitations, which are independent from the axes of the coordinate system. For some nodes of a system such as supported nodes even the choice of all components of displacement may be useful.

In contrast to the components of displacement vectors the exact value of member forces of the equilibrated system cannot be directly set. The magnitude of member forces must be adaptable to meet equilibrium. However, as the scale of chosen initial magnitudes remains in general unchanged, dependencies among members, like a classification into primary and secondary elements, can be defined. Besides, a direct influence on the resulting system is also gained by choosing the distribution of compressive and tensile forces in the initial system, as the sign of member forces remains unchanged unless external loads with strong tangential components are applied.

The determination of parameters, which are considered to be supernumerary, and the assignment of their values are in general arbitrary. Below two strategies for the choice of supernumerary parameters are discussed. It is presumed that the supported nodes of a system are fixed to their initial positions. The remaining number of supernumerary parameters then coincides with the number of members in the system.

A possible strategy for the choice of supernumerary parameters is to assume a projective relationship between the initial and the equilibrated system, which means that the displacements of all nodes are parallel. This is obtained by either setting two components of the displacement vectors of all nodes to zero or by defining two relations between the components of the displacement vectors. Possible sets of initial member forces, which can be considered as supernumerary additionally to the defined displacement directions, coincide with the member forces, which must be set to uniquely determine a force diagram of the initial system. Figure 2 shows an example of an equilibrated system generated by assuming a projective relation. To uniquely determine the force diagram (Fig. 2c) the magnitude of an initial member force of one segment out of each of the curves A to E (Fig 2b) must be chosen (member forces along curve F are assumed to be zero).

Another strategy for the choice of supernumerary parameters is to set the magnitudes of all initial member forces. This approach will always enable an equilibrated solution of the system as long as the magnitudes of at least two initial member forces per node are not zero. Even initial configurations, which violate the equilibrium conditions, will always result in an equilibrated system, as equilibrium can be found by displacement of the nodes. Thus, this strategy lends itself especially to systems with only a few supported nodes, as shown in figure 1a.

Figures 2 to 4 show different solutions of one and the same problem by choosing different sets of supernumerary parameters. All systems are subjected to the same vertical loads and the magnitudes assigned to the supernumerary initial member forces coincide for all cases. Assuming a projective relation allows the best control over the geometry of the resulting equilibrated system, as figure 2a illustrates. In contrast, the choice of all initial member forces leads potentially to deviations from the originally defined geometry as the difference

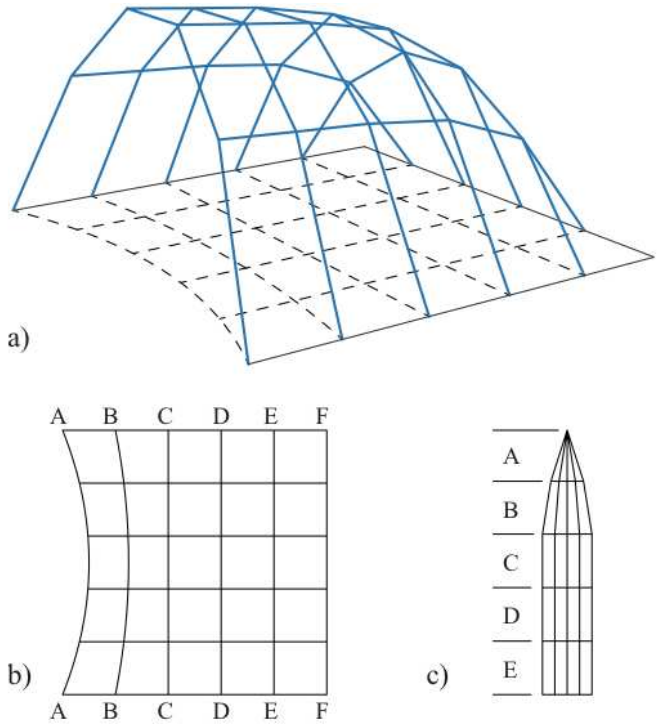

Fig. 2: a) Equilibrated system generated by using a projective relation; b) plan view of the initial system; c) force diagram of the initial member forces

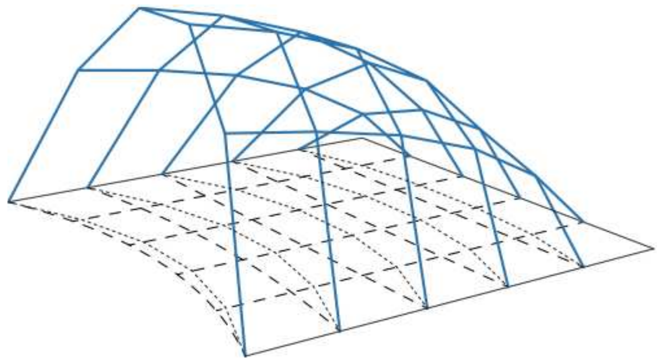

Fig. 3: Equilibrated system generated by choosing all initial member forces

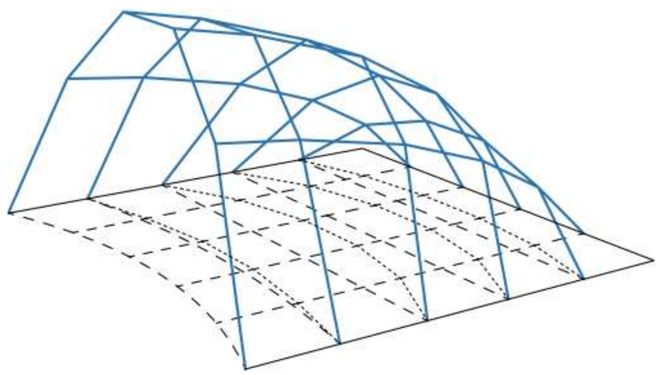

Fig. 4: Equilibrated system generated by choosing a combination of displacement components and initial member forces

between the initial system (dashed lines) and the horizontal projection of the equilibrated system (dotted lines) in figure 3 indicate. Considering the efficiency of load transfer the choice of all initial member forces leads to the best result for the exemplary system. Especially in the backmost part of the system the potential of biaxial load transfer is considerably better used than in the projected system. This results from the possibility of all nodes to displace freely, such that the most suitable form of the system for the assumed member forces is reached.

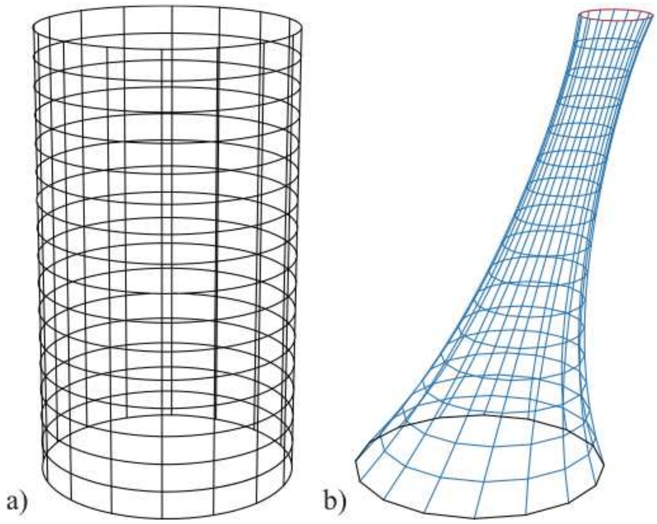

Fig. 5: Initial (a) and equilibrated (b) system generated by choosing a combination of displacement components and initial member forces

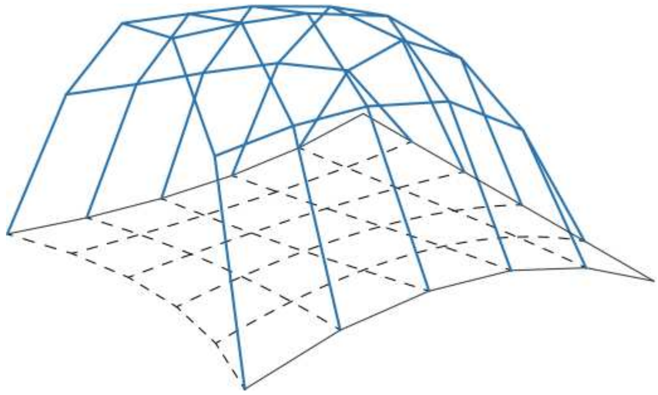

Fig. 6: Equilibrated system generated by using a projective relation based on a curved initial system

Besides the two discussed strategies it is also feasible to choose an arbitrary combination of parameters as supernumerary. Figure 4 illustrates a solution generated by using such a free combination of parameters. To assure that the horizontal projection of the unsupported edge of the equilibrated system does not deviate from the initial system, the horizontal displacement of the corresponding nodes has been fixed. In contrast, the remaining nodes of the system have not been limited in their displacement. Instead, the magnitudes of the initial forces of the members between them have been set to the same values as in the example shown in figure 3. Through the combination of a projective relation along the unsupported edge and the choice of all member forces in the remaining system the advantages of both strategies have been unified in the solution: control over the decisive geometrical parameters and an effective load transfer. A smart combination of parameters, which are considered to be supernumerary, allows the generation of complex equilibrated systems. Figure 5 shows a closed strut and tie network, which is subjected to a combination of dead and wind loads. It was generated by setting the z-components of displacement of all nodes to zero. This results in a limitation of the displacement of nodes to a horizontal plane. Additionally, the initial member forces of the horizontal rings were chosen. Besides the topmost ring, which was defined to be in tension, for the remaining rings compressive member forces were assumed.

When choosing displacement components of nodes, the scalar equilibrium conditions in the fixed displacement directions must be reachable to achieve solvability of the system of equations. Figure 6 shows an example of a system generated under the assumption of a projective relation between initial and equilibrated system. Compared to the example shown in figure 2 the initial system has been curved, such that the initial member forces cannot form an equilibrated state. However, as only the equilibrium condition in direction z is violated and equilibrium can be met in directions x and y, in which the displacement of nodes has been fixed, and equilibrated system could be generated.

2.4 Lateral Displacement of Loads

A lateral displacement of loads may occur, unless the displacement of nodes has been defined to be tangential to the line of application of the loads. This lateral displacement of external loads violates a basic convention of structural mechanics, as the displaced load case is not statically equivalent to its initial counterpart. Considering the case that the proposed method is used for form finding. Most external loads are dependent on the form of the structure. Dead weight is dependent on the surface area and wind is dependent on the actual height of the structure. As loads are dependent on the final form of a structure, magnitudes and points of application of loads will adapt to the change of form of the structure. Thus, the final form of a structure can only be found in an iterative process and a lateral displacement of the initially assumed loads cannot be avoided. In case of using the proposed method for the design of a structure with a given form, an iterative process must be used, which adapts the applied loads, such that the effects of lateral displacement of loads are compensated.

2.5 Computer based generation process

For the generation of equilibrated funicular systems a self-programmed Python [6] application is used, which is implemented in the 3D-Design-Software Rhinoceros 5 [7]. The input of the initial arrangement of nodes and members is done graphically in Rhinoceros 5. The Python application prompts further data like the position of supports and external loads as well as the choice of supernumerary parameters using the graphical input methods of Rhinoceros 5. The system of non-linear equations is created by the Python application and written in a file. The software Mathematica 8 [8] is used to determine the solution of the system of equations numerically. Mathematica 8 allows a direct control of its Kernel via command line, such that the Python application can start the computation process in Mathematica 8 and directly receive the computation results. Based on the results a graphical output in Rhinoceros 5 is created by the Python application.

3. Curved Stress Fields

Stress fields as described by Drucker [1] and Muttoni et al. [2] allow the consideration of planar structural elements subjected to in-plane loads. If a stress field is in addition loaded laterally, it must curve to meet equilibrium. A method to develop continuous Curved Stress Fields based on differential geometry has been developed [3]. The form of a continuous Curved Stress Field, which transfers loads in two predefined directions, is determined by a partial differential equation. Due to the generally limited applicability of differential equations, a numerical approach for the determination of Curved Stress Fields is proposed based on the generation of the form of funicular systems presented in the previous chapter.

3.1 Strut and Tie Networks and Curved Stress Fields

The member forces of a strut and tie network must be distributed to determine stresses. In the proposed method a constant distribution of stresses over the width between members is assumed. The resulting planar stress fields approximate a Curved Stress Field.

3.2 Internal Forces of Solid Structural Surfaces and Curved Stress Fields

A thrust line is the centre line of the resultant of the internal forces of an arch structure. Accordingly, a Curved Stress Field represents the resultants of the internal forces of a solid structural surface. The internal forces of a structural surface can be deduced from the stresses of the corresponding Curved Stress Field and the eccentricity of Curved Stress Field towards the centre surface of the structure. For shell-like structures a single Curved Stress Field is sufficient to represent the internal forces (figure 7 on top). Equations (4) to (8) express the relations between internal forces of a Curved Stress Field and a shell element.

$$n_y = n_{y1} \cos \sphericalangle(\overrightarrow{n_y}, \overrightarrow{n_{y1}}) \tag{4}$$

$$n_{yx} = n_{yx1} \cos \sphericalangle(\overrightarrow{n_{yx}}, \overrightarrow{n_{yx1}}) \tag{5}$$

$$v_y = n_{y1} \sin \sphericalangle(\overrightarrow{n_y}, \overrightarrow{n_{y1}}) + n_{yx1} \sin \sphericalangle(\overrightarrow{n_{yx}}, \overrightarrow{n_{yx1}}) \tag{6}$$

$$m_x = n^*_{y1} \cdot e_1 \tag{7}$$

$$m_{yx} = n^*_{yx1} \cdot e_1 \cdot \cos \sphericalangle(\overrightarrow{n_y}, \overrightarrow{n^*_{y1}}) \tag{8}$$

These equations apply analogously to the remaining internal forces. Considering structures like plates a second stress field, illustrated at the bottom of figure 7, must equilibrate the horizontal thrust, which is caused by a Curved Stress Field at the supports.

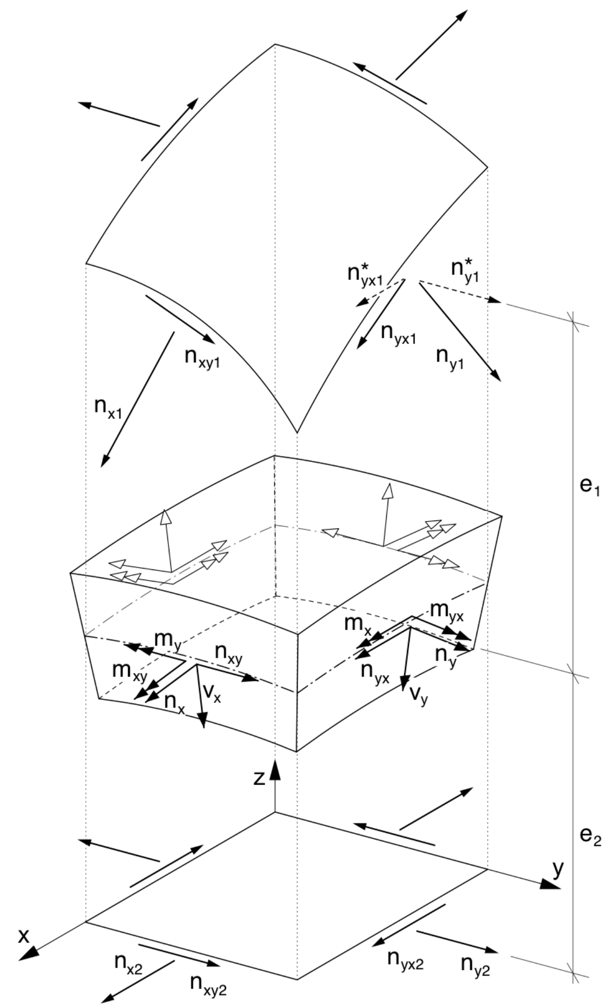

Fig 7: Relation between the internal forces of a Curved Stress Field (top) and an element of a surface structure (middle) and, in case of a plate-like structure, the additional stress field (bottom) equilibrating the horizontal thrust

Due to the holistic consideration of internal forces the distribution of forces in a solid structural surface can easily be understood. Accordingly, the possibilities to directly control load transfer in these quasi-infinitely statically indeterminate structures can be used intuitively.

3.3 Yield Stress Fields – The Limitation of Eccentricity of Curved Stress Fields

A Curved Stress Field and its eccentricity towards the centre surface of the structure represent the internal forces of a solid structural surface. To enable a comparison with the structural resistance a method is proposed, which describes the yield criteria graphically and avoids a conversion to internal forces. Based on the stresses of the Curved Stress Field and the decisive yield criterion an upper and a lower bound of the eccentricity of the Curved Stress Field can be determined. The bounds of eccentricity form two surfaces, which we denote as Yield Stress Fields. If a Curved Stress Field has exactly the form of one of the yield stress fields, all points of the structural surface will be yielding. As long as the curved stress field lies between the two yield stress fields, no point of the structure will be yielding, accordingly. This graphically based comparison allows a consideration of material strengths in the form finding process, such that a form does not have to coincide with the developed Curved Stress Field, but may differ from it as long as the Curved Stress

Field is enclosed by the two Yield Stress Fields. Yield criteria for structural surfaces made of reinforced concrete, which consider the relation of stresses of a Curved Stress Field and its eccentricity, are currently investigated.

4. Conclusion

A general method to create equilibrated strut and tie systems, which is applicable to any arrangement of nodes and members, has been developed. Based on this method numerically approximated Curved Stress Fields can be determined, which are used to describe the internal loads of laterally loaded solid structural surfaces. The combination of a Curved Stress Field with a stress field equilibrating the horizontal thrust at the supports is proposed to apply the method of Curved Stress Fields to structures like plates, which are purely subjected to flexural loads. Structural resistance is considered by Yield Stress Fields, which define the bounds of eccentricity of the corresponding Curved Stress Field.

Compared to Thrust Network Analysis [5] the presented approach to three-dimensional equilibrium of strut and tie systems extends the scope of such form finding methods to the entirety of network systems subjected to axial forces. As the range of possible applications is widened, the control of the generation of equilibrated strut and tie networks becomes less intuitive compared to Thrust Network Analysis [5]. However, provided a rough vision of the aspired form and some experience in the relationship of form and forces the proposed method represents an efficient tool to create equilibrated strut and tie systems.

Beyond the application of the proposed approach to three-dimensional equilibrium of strut and tie systems to curved surfaces, the method has even further potential. The defined general relation allows the determination of form and member forces of any system consisting of axially loaded members, planar and spatial.

Due to the holistic consideration of internal forces in Curved Stress Fields a great clearness in the illustration of the results is obtained compared to diagrams of the individual forces and moments. Besides the proposed direct determination, Curved Stress Fields can generally be used to just illustrate internal forces of structural surfaces. As there exists the presented direct relation between Curved Stress Fields and internal forces this benefit to the clearness of illustration is generally useable, irrespective of the method.

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