Geometry of Structural Form

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Abstract. *This paper describes a precise geometric method for the inscription of structural constraints into architectural form. Based on techniques from graphic statics, the force distribution in building structures is visualized using geometric diagrams. This diagrammatic representation allows a formal description that shows the relationship between the force flow and the structural form. The formal character of this description enables the direct implementation of a parametric truss geometry that maintains major structural behavioral characteristics under deformation. An interactive model of a structural freeform roof is developed through this link between a parametric truss definition and a design-driving NURBS surface. This allows for an intuitive exploration of the constrained design space in real time. Formal explorations and the comparison with built examples demonstrate the effectiveness of this approach.*

Keywords: Architectural Freeform Surface, Structural Geometry, Interactive Design, Reciprocal Diagrams, Parametric Modeling, Graphic Statics

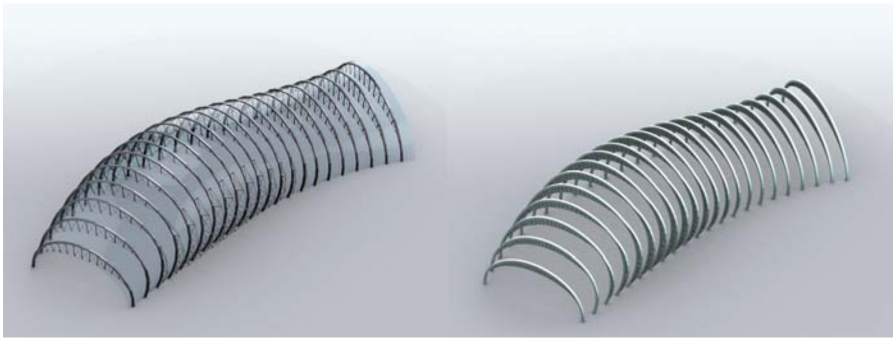

Figure 1: These freeform roof structures have efficient load-bearing capabilities and were designed by the direct integration of structural constraints into form. The methods used in the design of these structures are presented in this paper.

1 Introduction

Since the first use of digital design tools in architecture during the 1990s, the popularity of geometric modeling techniques has constantly increased. Originally coming from the airplane and film industries, these design tools were eventually adapted and embedded into architectural design software. In the last decade, the emergence of parametric modeling and scripting techniques in architectural CAD applications has enabled a new level of sophistication in freeform design.

With growing formal complexity in design, the realization of such forms becomes increasingly challenging. Substantial research has been done on the problem of geometric constraints of architectural freeform surfaces [Schober 2002], [Shelden 2002], [Pottmann 2007], [Pottmann 2008], whereas research on their structural behavior remains fragmentary. For this reason, it is necessary to develop strategies for the integration of structural constraints in the design of architectural freeform surfaces.

Structural limitations have to be integrated in the design process of freeform surfaces in order to reduce the amount of building material without decreasing the stability and usability of a structure. Methodologically, two different directions can be identified.

On the one hand, optimization approaches, that are limited to a post-rationalization of shape. They focus on using numerical methods to enhance a given structure, by minor changes in geometry, in a late design phase [Bollinger et al. 2005], [Sasaki 2007], [Tessmann 2008].

On the other hand, direct approaches, which integrate structural constraints into the design process in an early phase. The first attempts in this approach were based on physical models as a possibility for a direct integration of structural constraints with architectural form [Gass 1990], [Kotnik 2010]. The recent development of digital simulations of the physical behavior of hanging models has successfully integrated structural constraints in the digital design process of compression-only surfaces [Kilian 2004]. In direct approaches, the shape is the direct result of the flow of forces through the material. The visualization and construction of this force flow, therefore, can be seen as the starting point for the early integration of structural constraints into the design process.

Graphic statics is a geometric method for the representation of the force flow in structures that was developed in the mid-19th century [Culmann 1875], [Maurer and Ramm 1998]. This technique can be used to either analyze the internal forces in a structure for a given load condition, or to design a structural form for a given force distribution. As an example, the geometry of the Eiffel tower was designed using graphic statics [Charlton 1982]. The visual, diagrammatic character of graphic methods allows an intuitive exploration of different design alternatives and their structural implications [Muttoni 2004], [Schwartz 2009], [Allen and Zalewszky 2009].

Recent approaches combine graphic methods with interactive web applications for educational purposes [Greenwold and Allen 2001], [Block and Ochsendorf 2005]. The development of associative geometric modeling tools for architects, for instance *Grasshopper* and *Generative Components*, allows for a direct integration of geometric relations into an interactive parametric model.

The topic of this research is the integration of structural constraints in a parametric model using graphic statics. This paper is structured as follows: Section 2 briefly summarizes the geometric foundations of graphic statics. Section 3 shows a graphical design method for trusses. Section 4 reveals the concept of a structural relationship derived from the invariance of force-flow. To conclude, Section 5 describes the setup of the interactive parametric model of a freeform roof. The generation of examples of roof forms illustrates the power of this approach.

2 Graphic Statics and Reciprocal Diagrams

The method of graphic statics is based on the *reciprocal* relationship between two diagrams, that was first described for planar cases by Clerk Maxwell and Luigi Cremona [Maxwell 1864], [Cremona 1890]. Later, this reciprocal or *dual* relationship was extended to fully three dimensional cases [Crapo 1979]. One diagram, the *form diagram*, represents the geometry of the structure and the location of the applied loads. Its reciprocal figure, the *force diagram*, represents the vectorial force distribution in the structure [Muttoni 2004], [Schwartz 2009] [Allen and Zalewszky 2009]. In general, there is not a unique reciprocal diagram for a given form diagram. The force diagram is always scalable, it does not represent the absolute values of the forces, but rather the ratios between the forces. Irrespective of scaling, there is only one force diagram for a statically determinate structure. The question of degrees of freedom in the reciprocal diagram for indeterminate form diagrams has been studied recently [Block 2009]. Reciprocal figures have following properties:

- Each line L in the force diagram represents one dual line L^* in the form diagram and vice versa.
- Corresponding lines in form and force diagram are parallel.
- The length of the lines in the force diagram is proportional to the forces in the structure.
- If one connects the supports and the load vectors in the form diagram with an outer point, then the dual graph of this figure has the same topology as the force diagram. (Fig. 2)

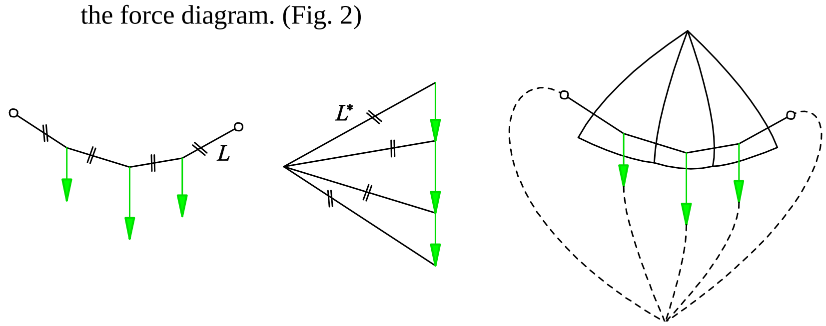

Figure 2: A *funicular polygon*: the form of a hanging string under the influence of forces, its force diagram and the visualization of their topological relationship.
The methods of graphic statics are applications of discrete differential geometry in two dimensions. For instance, in the construction of the *funicular polygon,* the form of a hanging string under the influence of forces corresponds to the repeated integration of the load vector field due to a graphical algorithm [Maurer and Ramm 1998]. With an increasing number of segments in a funicular polygon, the directions of the segments become tangents to a curve and its shape eventually attains that of the continuous curve of a string under the influence of a uniform load. (Fig. 3)

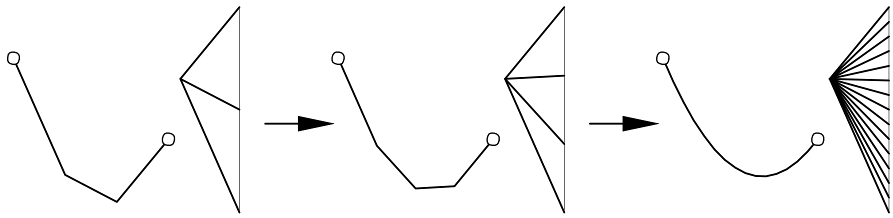

Figure 3: The funicular polygon leads to the form of a continuous curve of a hanging string with an increasing number of segments.

3 Geometry of an Efficient Truss

In order to demonstrate the method of graphic statics and its application to design, a geometric method for the procedural construction of a planar truss will be presented. The technique is based on a design method for the *constant chord force truss* [Allen and Zalewsky 2009]. This method generates a truss form with the top chord in pure compression and the bottom chord in pure tension for dead load[1]. Additionally, the tension forces in the bottom chord are all equal. While Edward Allen and Waclaw Zalewsky describe the application of this method for specific top chord shapes, this paper explores the possibility of this method for arbitrary top chord forms.

The truss form is constructed from a given discrete curve S consisting of the segments $S_1, S_2, \ldots, S_n$, defining the geometry of the top chord, and chord force F. For each node of the top chord, S, a dead load component, Q_i, is assumed. The first step is to construct the reciprocal diagram from the chord segments S_i , the nodal weights $Q_1, Q_2, \ldots, Q_{n-1}$,, and F. The second step is to construct the bottom chord of the truss.

The construction of the force diagram is straight forward: The nodal loads Q_i^* in the force diagram are graphically added. The support forces A and B are derived either by the lever rule or graphically by a trial funicular [Schwartz 2009]. The circle C is then constructed around the tip of the force vector A^*, with radius F.

[1] In structural design, permanent loads, especially the self weight of the building components, are called *dead loads*. All changing loads like wind, snow, movable objects, and people are called *live loads*.

The absolute value of F must be large enough such that the reciprocal load components Q_i^* are entirely located inside the circle. Next, construct the rays S_i^* in the direction of the top chord segments S_i. (see Fig. 3) The connecting lines between the intersection points $I_i = C \cap S_i^*$ between the circle and the rays are the reciprocal representations P_i^* of the truss members connecting the top and the bottom chord P_i. The representation of the force vectors W_i^* in the bottom chord are constructed by the connection of the intersection points I_i on the circle with the center of C. To construct the geometry of the bottom chord in the form diagram, start at support A and continue from left to right to the successive intersection of rays in the direction of P_i^* and W_i^*, which are the nodes of the chord.

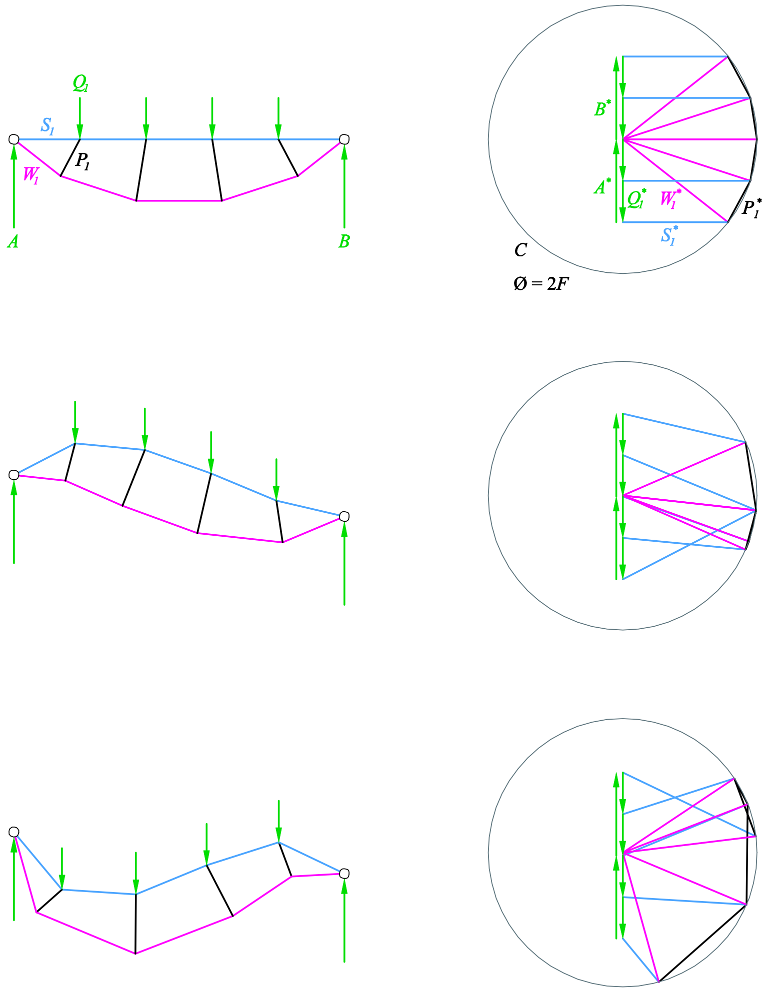

Figure 4: Form and force diagram for a constant chord force truss with five segments is shown in the first example. Implications of the deformation of the top

chord on the reciprocal diagram and the truss form can be seen in the two following examples.

4 Invariants in Force-Flow and Structural Classification

The above example of constant chord force trusses shows that the relationship between form and forces is fluid. However, there are properties that remain unchanged for transformations of the top chord S. These invariant properties enable a classification of truss geometries by similarities in the inner force distribution, independent of the morphological appearance of the truss.

The force diagram (Fig. 4) clearly shows that all dual representations of forces in the chord segments W_i are radii, thereby demonstrating that are all inner forces equal:

$$\overline{W_i^*} = F \quad (1)$$

The dual representations of all truss members in the force diagram are contained by the circle, so all inner forces in all truss members are less or equal the circle diameter:

$$\overline{S_i^*} \leq 2\,F \quad \text{and} \quad \overline{P_i^*} \leq 2\,F \qquad (2)$$

The forces in the top chord segments S_i and in P_i are all compression, while in the bottom chord the segments W_i are all tension. Together with (1) and (2), this allows a precise estimation of the dimensions of the truss elements, independent of the exact geometry of the truss.

For each transformation $f{:}\,S \rightarrow S'$ that maps the segmented curve S to a curve S', a mapping $g{:}\,T(S) \rightarrow T'(S')$ exists, which maps truss T to T'. Without changing the chord force, F, the relations (1) and (2) are invariant for g. These invariant properties of the force distribution in truss geometries allow a classification of structural behavior based on properties of the force-flow. Typical structural classifications are based on morphology, e.g. the terms “beam,” “arch,” and “frame” point to a specific shape, more than to a specific structural behavior. All truss geometries that fulfill (1) and (2) may be seen as different shapes of a common class of related truss geometries. (Fig 5.) The examples demonstrate that the integration of structural constraints results in a coupling of design parameters. This calls into question the traditional differentiation of structural systems by shape. This definition of structural classes, which is based on geometric invariants of the force diagram as opposed to similarities of the form diagram, opens up a new understanding of building structures and the continuous relationship among structural forms.

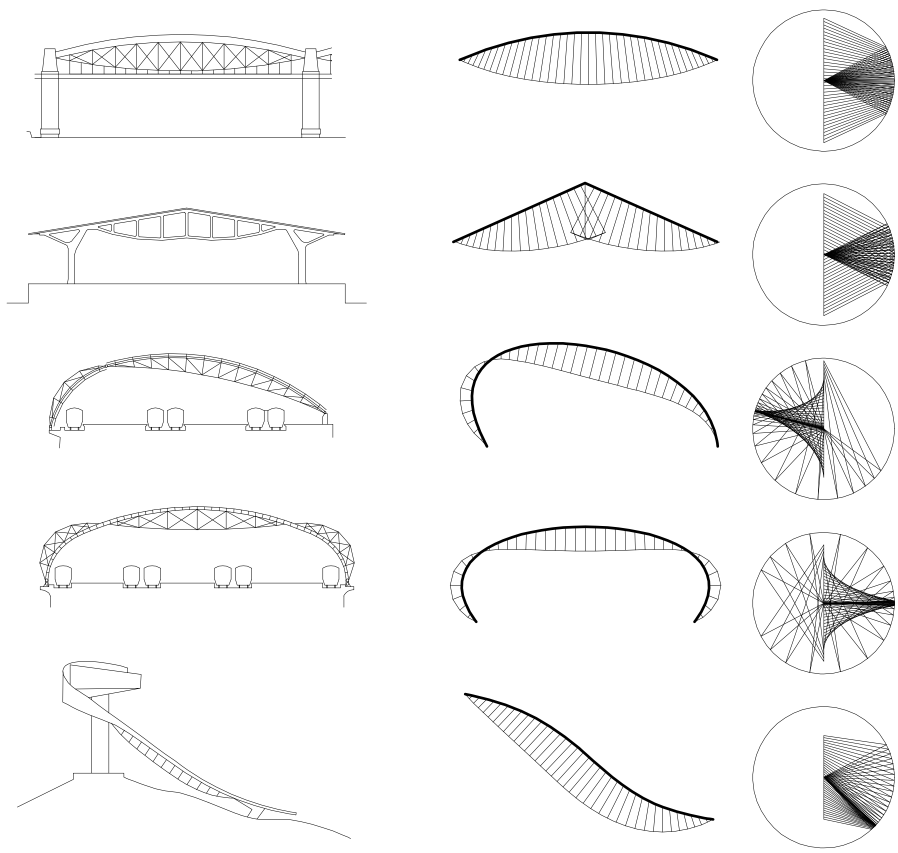

Figure 5: Building examples on the left, and examples of constant chord force truss geometries on the right. Their similarity in structural behavior becomes obvious in the force diagrams. From above: Luis Brunel: Royal Albert Bridge 1859; Robert Maillart: Chiasso Shed 1924; Grimshaw Architects: Waterloo Station 1993; Gerkan Marg + Partner: Lehrter Bahnhof 2002; Zaha Hadid Architects: Ski Jump 2001.

5 Parametric Model of an Efficient Freeform Roof Structure

This section describes an interactive parametric model of a freeform roof structure based on the above discussion. In the model, the gradually changing sections of the roof geometry are treated with the same structural principle without using the same truss morphology.

The parametric definition is built up using the associative modeling plug-in, *Grasshopper*, for the NURBS modeling CAD software, *Rhinoceros*. The definition has two input geometries: the freeform roof surface and a guiding curve in the XY plane. The positions of the trusses are given by the guiding curve that is divided in segments of equal length. Straight lines normal to g define the truss axes. The top chord geometries of the trusses are derived by the segmentation of the vertical section curves through the roof surface. The self weight of the structure at each node of a truss is calculated by a polygonal approximation of the neighboring area around the node. (Fig 6.) Additionally, the model has four numerical input parameters: number of trusses, segments per truss, chord force, and a scale factor for the dead load.

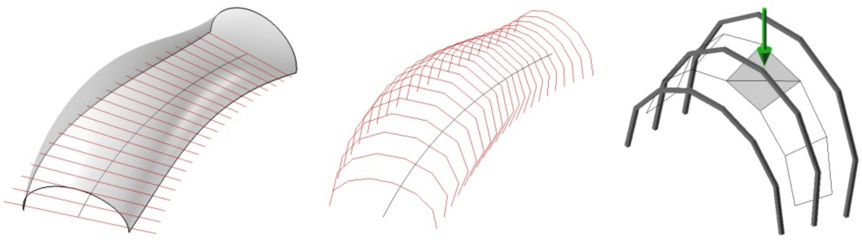

Figure 6: The generation of top chord geometry and loading from the input surface and guiding curve.

This parametric model enables the intuitive exploration of design alternatives for freeform structures supported by two edges. The interactive modification of input geometry works in real time because the direct procedural approach of geometry generation does not depend on computationally intensive algorithms. This allows for instant feedback to study the structural implications of formal choices. In the example of a roof model with 50 trusses and 50 segments, changes in the surface geometry are updated within approximately one second using an Intel Core Duo Processor with 2.8 GHz. The use of a NURBS curve and a surface as input geometries provides full freedom in the design process. The input surface can either be manually sculpted by dragging the control points or generated by another script or algorithm. (Fig. 7)

It must be noted that truss geometries generated by the method described in Section 3 are in equilibrium only for the designing dead load. In order to resist additional loads such as wind and snow, the trusses have to be further stiffened. This could be achieved by the installation of diagonal braces or by a stiff top chord girder. (Fig. 1)

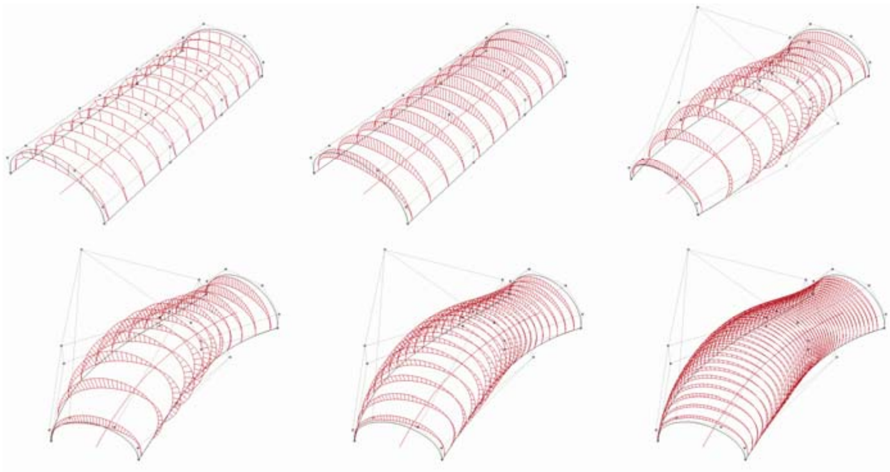

Figure 7: Design parameters of parametric roof are truss segmentation, truss number, surface form, and the form of the guiding line of the truss axes.

6 Conclusion and Future Work

As shown in this paper, the methods of graphic statics are easily applied to parametric CAD systems. These drafting-based techniques can be directly implemented using the rich variety of geometric operations provided by associative modeling environments. This allows for the direct integration of structural constraints in an interactive model that is based on the definition of geometric relationships and does not rely on additional computational techniques or software packages. Through graphic statics, structural problems can be treated with geometric means and may become questions of form.

This paper not only shows a new possibility for the development of interactive design tools, but also opens up a formal way of structural classification. The suggested classification is based on the inner force-flow and goes beyond the classical morphologic typologies of load bearing elements. It introduces a continuous relationship between structural forms, as it has been described a few years ago [Reiser and Umemoto 2006].

This is a first step towards parametric structural design based on graphical methods, shown for one specific class of truss geometries. Extensions are possible in several directions; a rich source may be historic collections of graphical methods [Malcolm 1914], [Wolfe 1921]. This work provides a basis for future exploration in the application of this method for different support conditions in two dimensions, such as a cantilevering beam a continuous beam. Another direction may be the extension of this truss to a spatial system, based on three dimensional extensions of the reciprocal relationship [Crapo 1979], [Micheletti 2008].

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