Discrete Analysis – A Method to determine the Internal Forces of Lattices

Thomas KOHLHAMMER
PhD Student
ETH Zurich
Zurich, Switzerland
kohlhammer@arch.ethz.ch

Thomas Kohlhammer, born 1974, received his civil engineering degree (2001) and his degree in architecture (2004) from the University of Stuttgart. His main area of research is related to lattice structures.

Toni KOTNIK
Senior Researcher
ETH Zurich
Zurich, Switzerland
kotnik@arch.ethz.ch

Dr. Toni Kotnik, born 1969, received degrees in architecture from the University of Utah and the ETH Zurich and degrees in mathematics from the University of Tubingen and the University of Zurich.

Summary

The following text shows the state of research in developing a method for structural analysis of planar and curved lattice systems, whereby the methodological approach refers to the structural behaviour of Reciprocal Frame Systems. The method is based on the consideration of equilibrium for the bar elements and their interacting forces in the joints, and thus it develops, according to plasticity theory, a possible equilibrium solution for the internal forces of the overall structure. Because of the independence of material properties, the analysis method will show the relationship between force and form, even of complex structures, in a simple and direct way.

Keywords: *discrete structures; bar-shaped structures; spatial structures; grid shells; lattices; Reciprocal Frame Structures; structural analysis; analysis method.*

1. Method of Discrete Analysis

With Discrete Analysis a descriptive tool is developed to determine the internal forces in planar and curved lattices, with focused consideration on free-formed structures. In contemporary architecture they gain an ever-increasing importance, however are statically difficult to control, in comparison to funicular forms. In a first step the methodology of the analysis was derived from the structural principle of Reciprocal Frame Systems (see Figure 1) and in a second step it will be generalized. Reciprocal Frame Structures refer to a discrete structural surface, which can be formed curved or planar. Built with short bar-elements they span many times the length of the individual bars. Although the connection nodes of the bars can be simple, i.e. in a static sense hinged, a rigid overall structure results from the specific arrangement of the bars.

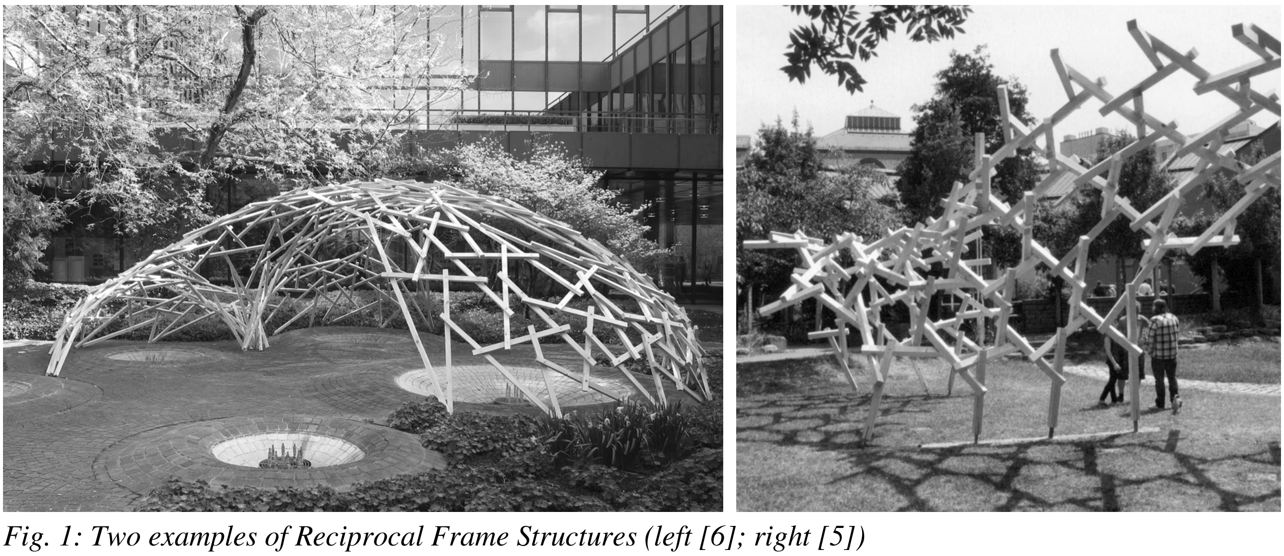

Fig. 1: Two examples of Reciprocal Frame Structures (left [6]; right [5])

The structural principle of Reciprocal Frames is that each individual bar in the system functions as a simple beam. This beam lies on another bar at each of the bar's ends, as well as it bears the supporting force of the one or two bars resting on it and optional dead loads or live loads. This forms reciprocal figures of interwoven elements that circumscribe polygons. The bearing of one another of the bar elements generates forces in the connecting nodes, which cause, due to the reciprocity of the system, a specific interaction of the elements within the overall structure. The Reciprocal Frame System appears to be suitable as a model system for an analysis method, since a significant structural aspect is the bending of the bars, and that is also an essential part in free-formed structural surfaces. In the following the development of the method on the basis of structural behaviour of Reciprocal Frame Systems is explained and afterwards the evolution to a generalization is outlined.

2. Methodology

The methodology of Discrete Analysis is based on the equilibrium of forces in the static subsystems of each individual bar. A load of it arises reaction forces in the connecting nodes to the neighbouring bars, where these forces turn acting. The methodology starts such a consideration in all loaded nodes and continues iteratively (see Figure 2). Here, in each iteration step, the reaction force in a connecting node is seen as an increasing of the nodal force. Due to the conditions of equilibrium in a single beam each reaction force of a subsystem represents a percentage of the acting force. Thereby the increases of nodal forces decrease with each iteration step and tend to zero. In every connecting node the sums of increases tend to a limit value, which represents a valid equilibrium solution in the system as a whole.

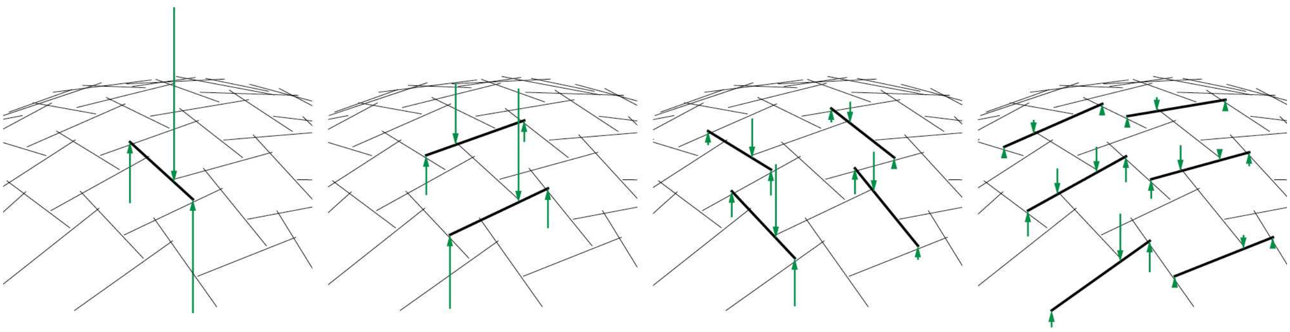

Fig. 2: Four steps of the iteration process

The individual bars are considered as simple beams with statically indeterminate supports, resulting in one degree of freedom for the equilibrium in each subsystem – the inclination angle of the reaction forces (see Figure 3). The degree of freedom can be eliminated either by the construction of the node connections, or, in the case where the compounds provide the maintaining of the static indeterminacy, by reasonable assumptions. The method of Discrete Analysis shows whether the chosen constitution to eliminate the degree of freedom in the sub-systems results in an equilibrium solution for the entire system.

Fig. 3: Two examples of inclinations of reaction forces in a statically indeterminate system (each left: form diagram; each right: force diagram)

The inclinations of the reaction forces in the connecting nodes have a significant impact on the behaviour of the overall structure. For example, if the bars just lie on each other, the compounds enable a rather small inclination caused by friction, thus the entire system obtains mainly flexural bearing capacity. However, if the node connections enable more inclined reaction forces, a higher in-plane bearing capacity of the overall structure can be achieved. In Discrete Analysis, the inclination of reaction forces is definable for each bar element and the method calculates, if existing, an equilibrium solution for the internal forces of the entire system. The instantaneous results of the methodological algorithm are the vectors of the nodal forces, which define the essential performance of construction of the connecting nodes. Furthermore the stresses of the bars are calculated out of the nodal forces, whereby their material and geometric requirements are defined.

Since Discrete Analysis is based on statically clearly ascertainable elements and the mechanisms of force transmission in the compounds are easily definable, it provides the basis of a descriptive analysis method. Similar methods, such as the Finite Element Method (FEM), are considerably complex. In FEM analysis calculations, the structure is divided into finite elements, for whose interfaces an equation system of forces, displacements and material constraints is formulated, out of which the stresses of the structure are determined. There, the internal forces are calculated according to elasticity theory, and in doing so the result is dependent on material properties.

Discrete Analysis is based on plasticity theory and provides a possible equilibrium solution for the internal forces, which represents a value for the ultimate load according to the Lower Bound Theorem. The result is independent of material properties, thus producing a more direct relationship between force and form. This is illustrative, which is an extremely important aspect, when regarding the applicability of the method as a design tool for complex structures in the dialogue between architects and civil engineers.

3. Equilibrium of Elements and Nodal Constraints

3.1 Different Equilibrium Systems of Bars

Generally speaking, two types of subsystems for the bars of a Reciprocal Frame Structure exist. A bar that bears just one neighbouring bar refers to an equilibrium system comprising three forces. This type is mainly found in the region of margins of Reciprocal Frames. A bar that provides a support for two neighbouring bars corresponds to an equilibrium system comprising four forces. The inner region of a Reciprocal Frame System mainly consists of those types of elements. As an idealization in the consideration of equilibrium, all bars are reduced to their system axis. This implies that all points of application of the forces are assumed lying on it, and the effectively existing offset of these points perpendicular to the system axis is ignored.

3.2 Three-Forces Equilibrium System

This subsystem is defined by the acting force $\boldsymbol{F}_2$ and the two points P_0 and P_3, where the reaction forces $\boldsymbol{F}_0$ and $\boldsymbol{F}_3$ apply (see Figure 4). The problem of equilibrium defines a planar system of forces constituted by $\boldsymbol{F}_1$, P_0, P_3, which provides a solution for any positions of points of application on the system axis. The functions for calculating the reaction forces are formulated as follows:

$$\bar{\boldsymbol{F}}_0 = \frac{a \cdot b \cdot l}{r} \cdot \left\| \boldsymbol{F}_2 \right\| \cdot \boldsymbol{s} - b \cdot \boldsymbol{F}_2 \qquad (1)$$

$$\bar{\boldsymbol{F}}_3 = -\frac{a \cdot b \cdot l}{r} \cdot \left\| \boldsymbol{F}_2 \right\| \cdot \boldsymbol{s} - a \cdot \boldsymbol{F}_2 \qquad (2)$$

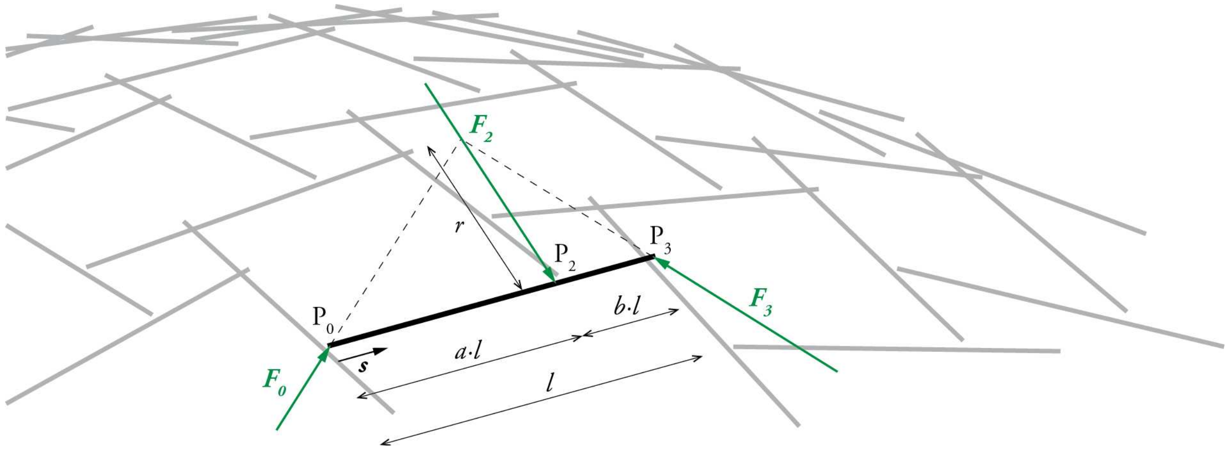

Fig. 4: Three-forces equilibrium system

As explained in Section 2, the inclination of the reaction forces is the value which has to be specified to eliminate the degree of freedom in the equilibrium system of the bars and which defines the proportion of flexural and in-plane bearing capacity of the overall structure. In the equations (1) and (2) this value is denoted by r, representing the distance between the intersection of all lines of application and the contact point of $\boldsymbol{F}_2$ on the bar. In structures with bending stress only, such as perpendicularly loaded planar Reciprocal Frame Systems, the acting and reaction forces in the subsystems are parallel ($r \rightarrow \infty$), so that the equations (1) and (2) simplify as follows:

$$\bar{\boldsymbol{F}}_0 = -\,b \cdot \boldsymbol{F}_2 \quad (3)$$

$$\bar{\boldsymbol{F}}_3 = -\,a \cdot \boldsymbol{F}_2 \quad (4)$$

Mathematically considered, the iterative process of Discrete Analysis describes a geometric series. An equilibrium solution for the internal forces of the overall structure exists if the series is convergent, which is given if the following condition is fulfilled:

$$\left\| \bar{\boldsymbol{F}}_\mathrm{k} \right\| \rightarrow 0 \quad (5)$$

Concerning equations (3) and (4), which refer to a perpendicularly loaded planar System, condition (5) is always fulfilled since $a < 1$ and $b < 1$. A detailed explanation of the special case of planar Reciprocal Frame Structures can be found in [2]. In the general case, described by equations (1) and (2), the fulfilment of the condition for convergence (5) is additionally dependent on the scalar factor r and the vectorial term $\boldsymbol{s}$. This means that the choice of the value that eliminates the degree of freedom affects both the type of bearing capacity and the convergence of the method, which is in turn equivalent to the existence of an equilibrium solution for the overall structure.

3.3 Four-Forces Equilibrium System

This type of bar, including two acting forces, represents a spatial system of forces, which is defined by the forces $\boldsymbol{F}_0$ to $\boldsymbol{F}_3$ and their points of application P_0 to P_3 on the system axis (see Figure 6). Generally speaking, in spatial systems of forces, even the existence of equilibrium is dependent on the given parameters. However, it can be shown that one exists, if a straight line can be found, which intersects the lines of application of all four forces. This is the decisive reason why the nodes are all assumed lying on the system axis of the bar and therefore it represents such a straight line. In further research, the impact of the idealization has to be explored in detail, respectively a way without establishing this idealization has to be found.

$\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are acting forces in this system, whereby there are two reasonable alternatives to set the supporting points. On the one hand only P_0 and P_3 at the bar-ends and on the other hand an additional support at P_1 or P_2. In both cases, the reaction forces can be determined separately as a result of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ and then be superposed. In the first case, the calculation of the reaction forces works with the functions shown in Section 3.2 as it is the superposition of two planar systems of forces which intersect at the bar axis P_0P_3. Here, the system has one degree of freedom. In the second case, the connecting nodes at points P_1 and P_2 are constructed in a manner that they can also function as a support. This is especially appropriate if a high in-plane bearing capacity of the overall structure is to be achieved, as for the in-plane forces a more direct load path can be formed (see Figure 5).

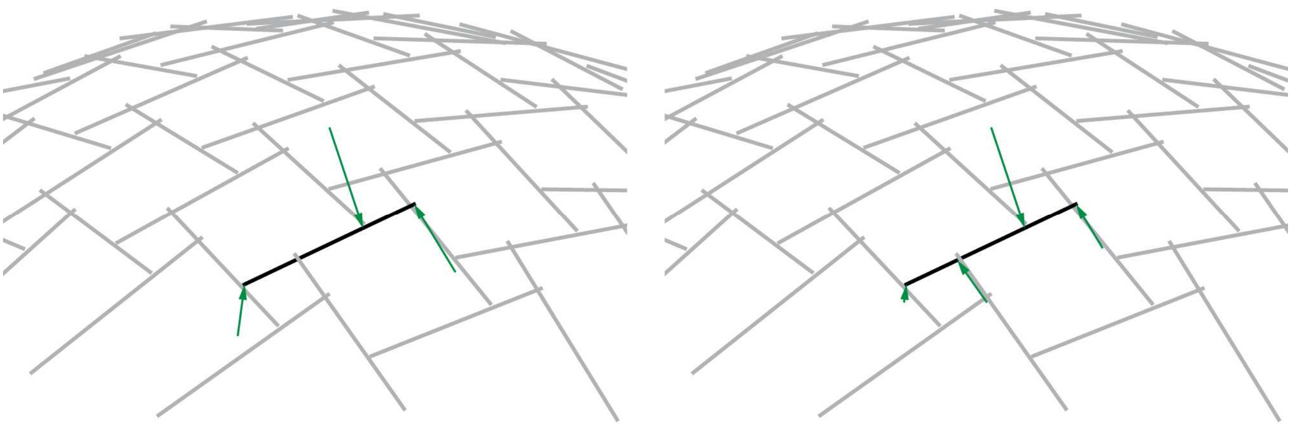

Fig. 5: Two types of load paths

The implication of this case is that, for the calculation of the reaction forces, there are no longer two planar systems with three forces each to be superposed but two spatial ones with four forces each. The spatial equilibrium system possesses two degrees of freedom. In this examination they represent again the inclination of the reaction forces at the bar ends and additionally the impact of the third support.

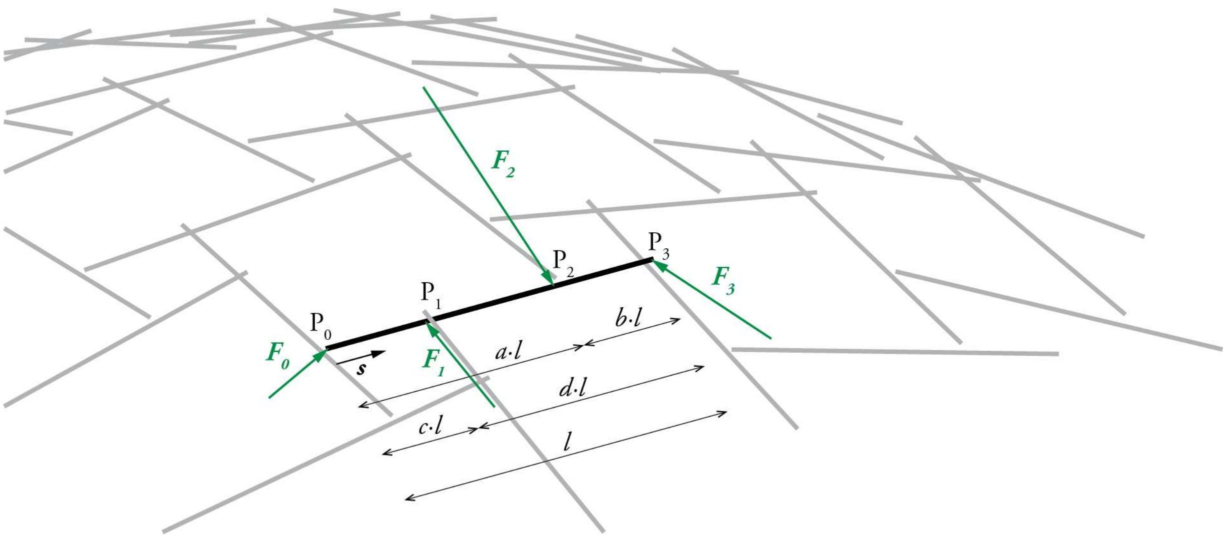

Fig. 6: Four-forces equilibrium system

In the following, the assumption is made that the vector of the reaction force at the third support point has the same orientation as the neighbouring bar, connected in this point. The parameter that eliminates the additional degree of freedom defines the proportion of the reaction force at this support, in relation to the other two. This reaction force can be determined by assuming a reasonable value of proportion.

Regarding that part of the superposition of reaction forces, in which $\boldsymbol{F}_2$ is acting and P_1 is considered as the third supporting point, the functions for calculating the reaction forces are formulated as follows. The same applies to the other part of the superposition in which $\boldsymbol{F}_1$ is the acting force and P_2 represents the third supporting point.

$$\boldsymbol{F}_0 = -\,d\cdot\boldsymbol{F}_1 + \bar{\boldsymbol{F}}_0 \qquad (6)$$

$$\boldsymbol{F}_3 = -\,c\cdot\boldsymbol{F}_1 + \bar{\boldsymbol{F}}_3 \qquad (7)$$

4. Influence of Parameters

4.1 Parameter Types of Discrete Analysis

As explained in Section 3, the degrees of freedom of the system are eliminated by the shown assumptions for the conditions of the reaction forces at the connecting nodes, which are defined as nodal parameters in the method of Discrete Analysis. They control the structural behaviour, particularly the proportion of flexural and in-plane bearing capacity. The ratio of the node distances in relation to the longitudinal direction of the bar is a basic geometric property of Reciprocal Frame Structures. This results in the compositional way the bars are assembled to the overall system. In Discrete Analysis the ratios of distances between nodes are defined as system parameters. In the following, the impact of the nodal and system parameters on the properties of the structure is discussed in more detail.

The distances between the nodes perpendicular to the system axis of the bars are also geometric parameters. They control the curvature of the structure indeed, but according to the idealization defined in Section 3, have no instantaneous impact on the structural behaviour of the system.

4.2 Nodal Parameters - Control of Structural Behaviour

The nodal parameters can be assessed in two ways. On the one hand, by selecting an appropriate type of construction for the connection nodes, which produce statically determinate subsystems and thus the inclinations of the reaction forces are defined. On the other hand, if the constructions for the connections maintain the redundancy of the subsystems and thus undetermined inclinations of reaction forces, the nodal parameters are to be reasonably assumed by the user of the method.

In the first case, the structural behaviour, i.e. the proportion of flexural and in-plane bearing capacity, is exactly determined. For this, the method provides both, the verification whether the appropriate type of construction for the connections allows an equilibrium solution of the overall system, and the vecorial terms of the nodal forces, so that the compounds and the cross-sections of the bars can be designed accordingly.

In the second case, the structural behaviour of the system is undetermined. Therefore the user can determine a particular one by reasonable assumptions for the nodal parameters. According to the users' determination, the method of Discrete Analysis verifies the existence of an equilibrium solution and calculates the corresponding nodal forces. With this, parameter variations can be made, and thus the performance of the structure relating to flexural and in-plane bearing capacity within plasticity theory can be explored. As shown in section 3.2, defining the location of the intersection of forces in the subsystem, represents an adequate way of placing a nodal parameter in the formulas of equilibrium analysis of the bars.

As an example, figure 7 pictures two solutions of Discrete Analysis, regarding a simple Reciprocal Frame Structure under dead loads. Figure 7 a) shows the resulting nodal forces relating to the assumption that the reaction forces at the compounds only have little inclinations - for example due to marginal friction at the contact points - and thus the overall structure possesses mainly flexural bearing capacity. Figure 7 b) shows the nodal forces which relate to the assumption that, due to an appropriate construction of the compounds, the reaction forces in the subsystems can be strongly inclined and thus the entire system possesses high in-plane bearing capacity.

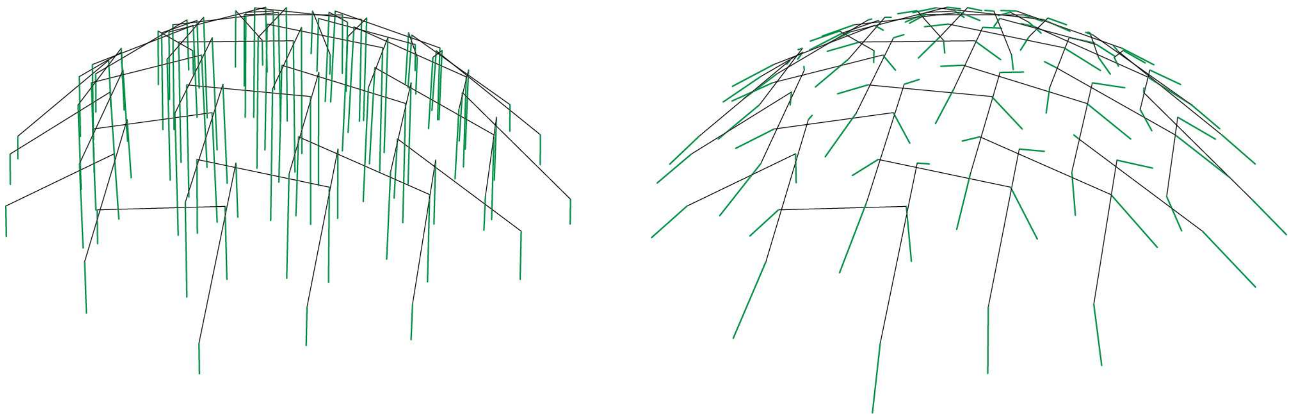

Fig. 7: Nodal forces calculated with Discrete Analysis (left: a) high flexural bearing capacity; right: b) high in-plane bearing capacity)

4.3 System Parameters - Control of Systemic Characteristics

Comparing the Reciprocal Frame System with other types of lattice structures, for example with a simple grid with pin-joined elements, some analogies may be identified. By cyclically dissolving the nodes of a pin-joined grid, where bars are only connected at the ends, a smooth transition to other related systems can be produced (see Figure 8). In Discrete Analysis, the dissolving of the nodes can be expressed by varying the system parameters in the functions of equilibrium conditions (1), (2), (6) and (7), which means a modification of the relative nodal distances *a*, *b*, *c* and *d*.

This allows the definition of a continuous spectrum of lattice-like structural surfaces, which includes, in addition to those already mentioned, other systems, such as the Zollinger-System. All members of this spectrum can be transformed into each other with smooth transitions.

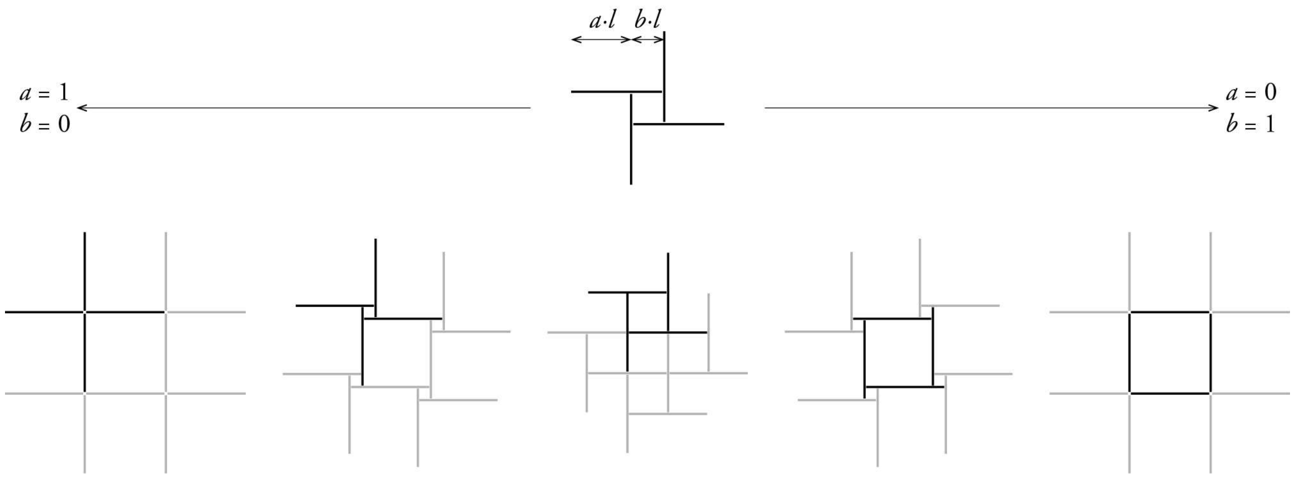

Fig. 8: Defined spectrum of lattice-like structural surfaces; left and right: pin-joined grid, middle: Zollinger-System, between: Reciprocal Frame System

With Discrete Analysis it can be illustrated, that the variation of system parameters also has a significant impact on the internal forces of the bars and on the nodal forces and thus on the potential of bearing capacity of the entire system. For example, the Reciprocal Frame System and the Zollinger-System provide flexural bearing capacity by using simple constructions of compounds with relatively little nodal forces. In contrast, lattices with pin-joined elements provide rigidity only with sophisticated knots stressed by high nodal forces.

5. Conclusion

The presented text showed the state of research in developing a method of analysis that is applicable to different types of lattice-like structural surfaces. The intention is to develop out of this a descriptive tool that determines the internal forces of complex structures on the basis of manageable elements and their interactions with each other.

Here, the approach should not be to solve the statically indeterminate systems by material constraints and the elasticity theory, but to find a possible equilibrium solution within plasticity theory by establishing reasonable assumptions to eliminate the redundancy of the system.

The outlined proceeding of dissolving the nodes, as well as the system parameters in the functions of the equilibrium analysis, showed the relationship of different lattice structures, which have become established as typologies such as Zollinger-System or Reciprocal Frame Structure. In the illustrated consideration, a spectrum with smooth transitions has been defined, which includes all these systems. Thus, the typological difference between lattice-like structural surfaces disappears, since each can be classified into the defined topological spectrum.

6. References

[1] BAVEREL O., DOUTHE C. and CARON J.F., "Nexorade: A Structure for 'Free Form' Architecture", Adaptables – *International Conference on Adaptable Building Structures*, Eindhoven, 2006

[2] KOHLHAMMER T. and KOTNIK T., "Systemic Behaviour of Plane Reciprocal Frame Structures", *Structural Engineering International SEI*, Vol. 21, No. 1, 2011, pp. 80-86.

[3] MUTTONI A., SCHWARTZ J. and THÜRLIMANN B., *Bemessung von Betontragwerken mit Spannungsfeldern*, Birkhäuser, Basel, 1997

[4] POPOVIC LARSEN O., *Reciprocal Frame Architecture*, Architectural Press Elsevier, Oxford, 2008.

[5] PROLL M. and GÜNTHER A., "Selfsupporting Framework", *Bauwelt*, Vol. 102, No. 1-2, 2011, p. 57.

[6] SPIRO A., THOENNISSEN U. and WERENFELS N., *Objects in Mirror are closer than they appear*, Research Project, ETH Zurich, Zurich, 2011

[7] WERENFELS N., *Digitale Weiterentwicklung des Hebelstabsystems*, Master's Thesis ETH Zurich, Zurich, 2009